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Challenging mathematics and its role in the learning process

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Challenge is not only an important component of the learning process but also a vital skill for life. People are confronted with challenging situations each day and need to deal with them. Fortunately the processes in solving mathematics challenges (abstract or otherwise) involve certain types of reasoning which generalise to solving challenges encountered in every day life. Mathematics has a vital role in the classroom not only because of direct application of the syllabus material but because of the reasoning processes the student can develop.

ICMI has commissioned a study to investigate the issues of challenge in the learning process. The speaker is one of the co-chairs of this study, which is scheduled to have its Study Conference in 2006 and issue its findings as a Study Volume in 2008. In this talk Peter will describe some of the attempts to define the concept of challenge itself and discuss various related issues which are being identified in the context of challenge and the learning process.

Hanna Neumann

First I would like to briefly comment on the significance of Hanna Neumann. Many of us here got to know B. H. Neumann well over the last thirty years and the pervasive positive influence he had on mathematics in this country. Hanna died much earlier and less of us who are here now knew her directly.

I did not know her personally but I was fortunate to have once attended an inspirational talk by Hanna, in Adelaide in about October 1971, only two or three months before she died. As usual, Bernhard sat proudly in the front row. My fortune to have been able to attend that Seminar helped me to understand the unique role that Hanna had in the inspiration of mathematicians and students in Australia. From the colleagues I know who were fortunate to work or study with her there is no doubt about the significant love she had for teaching and students and influence she had on the standards of teaching. It is a great honour to present this lecture named after her.

ICMI and ICMI studies

As I assume is well-known to this audience, the International Commission on Mathematics Instruction (ICMI) is the principal professional body for mathematics education and it conducts its activities in a number of ways. It publishes bulletins, holds an international conference every four years known as ICME (the most recent one having been held in July 2004 at Copenhagen), and is the umbrella body for five affiliated study groups (with themes of history, psychology, women in mathematics, competitions and most recently the group on applications and modelling).

ICMI also administers studies, which investigate particular issues with respect to mathematics education. Each study focuses on a particular topic in mathematics education, attempts to identify issues and to address them. Each study is chaired by a single chair or two co-chairs and an international group, known as the International Program Committee (IPC) of about twelve people is appointed to control the study.

The IPC meets fairly early in the process and initiates a Study Discussion Document, which identifies the issues and announces the program. Eventually a definitive conference for the Study is held. This Study is attended by invitees after the Study Volume has published the Study Document. Usually 70 to 100 people attend, but this is by invitation after people read the Discussion Document and show how they can contribute. It is not possible to attend the Study Conference as a passenger.

Finally a definitive Study Volume appears at the culmination of the Study. The whole process is likely to take about six or seven years, but the final Study Volume becomes an authoritative document, giving the state of the art after much input and discussion.

Past studies

At this stage fourteen studies have been completed. They started in the 1980s and the completed studies essentially cover the topics of

1. The Influence of Computers and Informatics on Mathematics and its Teaching
2. School Mathematics in the 1990s
3. Mathematics as a Service Subject
4. Mathematics and Cognition
5. The Popularisation of Mathematics
6. Assessment in Mathematics Education
7. Gender in Mathematics Education
8. What is Research in Mathematics Education and What are its Results?
9. Perspectives on the Teaching of Geometry for the 21st Century
10. The Role of the History of Mathematics in the Teaching and Learning of Mathematics
11. The Teaching and Learning of Mathematics at University Level
12. The Future of the Teaching and Learning of Algebra
13. Mathematics Education in Different Cultural Traditions: A Comparative Study of East Asia and the West
14. Applications and Modeling in Mathematics Education.

Present studies

There are three studies in progress. These are

15. Teacher Education and Development
16. Challenging Mathematics in and beyond the Classroom
17. Technology Revisited.

International Program Committee (IPC)

This Study is being co-chaired by myself and Ed Barbeau, of the University of Toronto. The Study Conference will be held in Trondheim during 2006 and the Chair of the Local Organising Committee is Ingvill Stedøy. Other members of the International Program Committee are Mariolina Bartolini Bussi (Italy), Albrecht Beutelspracher (Germany), Patricia Fauring (Argentina), Derek Holton (New Zealand), Martine Janvier (France), Vladimir Protasov (Russia), Ali Rejali (Iran), Mark Saul (USA), Kenji Ueni (Japan), and Bernard Hodgson (Canada), who is Secretary-General of ICMI. In addition two members of the ICMI Executive, Maria de Losada (Colombia) and Petar Kenderov (Bulgaria) have joined the IPC.

The members of the IPC have a broad range of activities. Some are involved in competitions or related activities while others are noted for involvement in exhibitions and school based projects which provide enrichment.

Timetable of the Study

The Study and the composition of the IPC were announced in early 2003 and the IPC met formally in Modena, Italy in November 2003 in order to commence the writing of the Discussion Document and plan the Study. Further work on the Discussion Document continued until its final text was agreed at a meeting of the IPC at ICME-10. The Document outlines the remainder of the program. There will be a Study Conference in Trondheim from 28 June to 2 July 2006. Applicants to attend this conference must apply to the co-Chairs by 31 August 2005, showing how they can contribute to the Study. It will not be possible to attend without contribution and it is expected that maybe up to 100 participants will be invited. Eventually by 2008 there will be a Study Volume published. This will be the formal outcome of the Study. It might be the Proceedings of the Study Conference, or it might be articles rewritten but inspired by papers at the conference.

Discussion document

This document occupies the central core of the Study. It is available to read at the Study website at www.amt.edu.au/icmis16.html. It comprises five chapters as follows.

1. *Introduction*
   This chapter basically describes what ICMI Studies do in general and the general aims of this study.
2. *Description*
   This chapter asks for the definition of 'challenge', asks how we are providing challenge, and where, with some brief examples.
3. *Current context*
   This is the longest chapter and goes much deeper into the use of challenge, listing a broad range of different types of use of challenge.
   These examples are classified and range from competitions (inclusive, exclusive, team, etc.), exhibitions, problem solving in schools, research activities, etc.
4. *Questions arising*
   This chapter lists many questions which arise from the previous discussion and which are asked here, prompting specific paths for the study.
5. *Call for papers*
   Finally the timetable and the method of participation are outlined.

Material available and scope of the study

The World Federation of National Mathematics Competitions (WFNMC), one of the five Affiliated Study Groups of ICMI, has published a policy document on similar matters to those which might be explored by the Study. In this document, to be found on the WFNMC website www.amt.edu.au/wfnmc.html.

This policy document defines the interests of WFNMC well beyond that of competitions, including a number of areas such as enrichment course work, maths clubs, research activities, publications, etc. In the last few years support for teachers has also become an increasing theme.

There are four other ASGS, and each has been closely associated with an ICMI Study. This Study is designed to go well beyond the areas beyond the WFNMC policy document and identify all areas in which mathematics challenge applies. Many members of the IPC are from quite non-competition backgrounds.

What is challenge?

There was considerable debate about this question and there will probably be no definitive answer. However it will become in itself one of the central areas of discussion in the Study.

Probably the most common definition will be based around the idea that challenge is the experience of meeting a new, unforeseen situation and coming to grips with it. This is a critical idea which I will pursue shortly. In the real world we continually have to face new situations and deal with them. It can be argued that by learning to meet challenge in a mathematical situation students will develop the powers of abstract reasoning that will enable them eventually to be able to systematically face other situations without apparent mathematical context.

Another proposal defined challenge as a non-traditional learning experience. Yet another defined it as the process of lifting oneself from one knowledge state to a higher one.

In and beyond

The Study has the words ‘in and beyond’. As such it will need to address issues in the classroom, presumably normally within the syllabus, and those many programs outside the classroom, and how each of these may affect the learning and teaching process. I will touch on some aspects.

In the classroom

To take up the first definition of ‘challenge’ I have posed, some will argue that in order to assist students in the learning process, the syllabus in various countries has become progressively defined to a higher level. As a result more specific outcomes are usually listed, placing time pressures on room for challenging mathematics.

There is also a question as to whether the topics which are in the syllabus lend themselves to challenge. Topics such as arithmetic and algebraic techniques, calculus and increasingly statistics, which dominate the syllabus, do lend themselves to challenge and problem solving situations.

How this is done, how it can be done better, and in fact answering the question of why it should be done, will be taken up in the Study.

Beyond the classroom

I will spend more time discussing the less familiar cases of challenge beyond the classroom. This ICMI Study acknowledges the growing demand to provide what we might call enrichment, complementing the syllabus with challenging and stimulating material, helping students to think mathematically, and from this experience develop a problem solving ability which can be helpful for broader life skills (such a demand is met by many 'suppliers' with the approval of the teacher).

Being able to do these things in a mathematical context arguably provides the ability to do so in other contexts.

This is an important aspect of mathematics. Too often in assessing potential syllabus material we can look too closely for direct application of a particular idea or skill.

All too often there are very good techniques in mathematics available to help in problem solving which are not in the syllabus, but which are nevertheless accessible to many students and improve the student's reasoning powers..

Internationally there are many examples of enrichment beyond the set syllabus, normally on a voluntary basis as some students seek to extend their knowledge and broaden their base for further study. The following topics in mathematics are often introduced to students in such enrichment situations and can lend themselves to student access quite easily:

- counting methods;
- pigeonhole principle;
- other methods of proof, e.g., contradiction, induction, invariance;
- discrete optimisation;
- geometry.

I give a few examples of where such methods have been used in problems posed to students. The problems below illustrate mathematical techniques which are used in enrichment programs are drawn from questions either set in the Australian Mathematics Competition for the Westpac Awards or the International Mathematics Tournament of Towns.

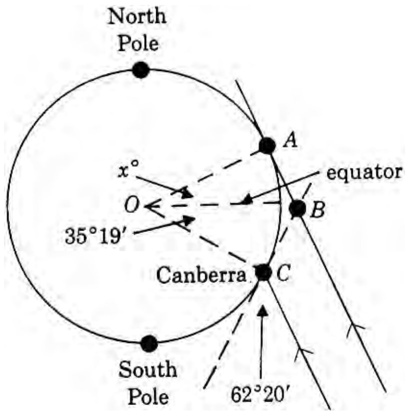
Problem 1 (geometry)

The latitude of Canberra is 35° 19' S. At its highest point in the sky when viewed from Canberra the lowest star in the Southern Cross is 62° 20' above the southern horizon. It can be assumed that rays of light from this star to any point on the earth are parallel. What is the northern-most latitude from which the complete Southern Cross can be seen?

Comments on solution 1

It is reasonably well-known that the Southern Cross can be seen in some northern latitudes. It was also used for navigation by the French aviators pioneering air rotes across the Sahara to South America.

The solution is quite accessible to secondary students familiar with line and circle geometry and uses the following diagram. The proof requires a cyclic quadrilateral.



Problem 2 (counting)

In how many different ways can a careless office boy place four letters in four envelopes so that no one gets the right letter?

Comments on solution 2

There are two ways of doing this. The intuitive way is to systematically list all of the cases. Numbering the envelopes as 1, 2, 3 and 4 the letters can be placed in the following orders giving an answer of 9.

2143, 2413, 2341, 3142, 3412, 3421, 4123, 4312, 4321

As with other counting methods this gives the teacher the opportunity to discuss generalisations, and this generalises to the derangement formula as discussed in Niven (1965, p. 80). This idea can be further generalised as follows.

Problem 3 (counting)
In the school band, five children each own their own trumpet. In how many different ways can exactly three of the five children take home the wrong trumpet, while the other two take home the right trumpet?

Comments on Solution 3
It is not too difficult to count the cases here either on intuitive reasoning.
Suppose the students taking home the wrong trumpet are called *A*, *B* and *C*.
These can take the wrong trumpets in two ways: e.g., *A* takes trumpet *B*, *B* takes trumpet *C* and *C* takes trumpet *A*, or *A* takes trumpet *C*, *B* takes trumpet *A* and *C* takes trumpet *B*.
We need also to know how many ways *A*, *B* and *C* can be chosen from the five. This is the same as the number of ways in which the two with the right trumpets can be chosen, this being ten; e.g., if the students are called *A*, *B*, *C*, *D* and *E* these are

A and *B*, *A* and *C*, *A* and *D*, *A* and *E*,
B and *C*, *B* and *D*, *B* and *E*,
C and *D*, *C* and *E*, and
D and *E*.

Thus the answer is $10 \times 2 = 20$.

Again this can be generalised to give a formula enabling the problem to be solved with any number of students and any fixed number to have the wrong trumpet.
The formula is

$$\binom{n}{r} \times D(r)$$

For $n = 5$ the solutions can be tabulated as

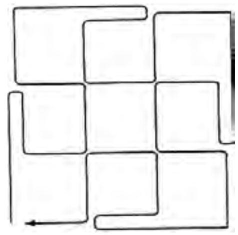
r	$\binom{5}{r}$	$D(r)$	$\binom{5}{r} \times D(r)$
0	1	1	1
1	5	0	0
2	10	1	10
3	10	2	20
4	5	9	45
5	1	44	44

with n being the number of students, r being the number of wrong trumpets.

There are many other counting scenarios which can similarly be invoked in real problems. These can lead to the inclusion exclusion principle, necklace counting method of Polya, etc.

Problem 4 (discrete optimisation)
A village is constructed in the form of a square, consisting of 9 blocks, each of side length l, in a 3×3 formation. Each block is bounded by a bitumen road. If we commence at a corner of the village, what is the smallest distance we must travel along bitumen roads, if we are to pass along each section of bitumen road at least once and finish at the same corner?

Comments on solution 4
The diagram shows a closed tour of length 28 and we claim this to be a minimum.



Each of the four corners is incident with two roads and requires at least one visit.
Each of the remaining twelve intersections is incident with three or four roads and requires at least two visits. Hence the minimum is at least $4 + 12 \times 2 = 28$.

Problem 5 (proof contradiction)
There are 2000 apples, contained in several baskets. One can remove baskets and/or remove apples from the baskets. Prove that it is possible to then have an equal number of apples in each of the remaining baskets, with the total number of apples being not less than 100.

Comments on solution 5
This problem looks very difficult but by assuming the result to be false we can more easily find a contradiction as follows.
Assume the opposite: then the total number of baskets remaining is not more than 99 (otherwise we could leave 1 apple in each of 100 baskets and remove the rest).
Furthermore, the total number of baskets with at least two apples is not more than 49, the total number of baskets with at least three apples is not more than 33, etc.
So the total number of apples is not more than $99 + 49 + 33 + \ldots$ This number is less than 2000. We thus have a contradiction.

Problem 6 (pigeonhole principle)
Ten friends send greeting cards to each other, each sending five cards to different people. Prove that at least two of them sent cards to each other.

Comment on solution 6
Dirichlet is reported to have first articulated this method of proof, which also bears his name, which is quite intuitive. If you have n pigeon holes and more than n pigeons to put them in, one pigeonhole must contain at least two pigeons. This proof essentially goes as follows.
We are given that each friend sends a card to 5 different of the other 9 friends. This means that there are $10 \times 9 = 90$ different routes.
By symmetry, these consist of 45 pairs (friend i to friend j and friend j to friend i).
However the number of cards sent is $10 \times 5 = 50$.
Since, each of these 50 cards is sent on a different route, by the pigeonhole principle at least $50 - 45 = 5$ cards must be sent in opposite directions along a repeated route, enough to prove what is required.

Problem 7 (invariance)

On the island of Camelot live thirteen grey, fifteen brown and seventeen crimson chameleons. If two chameleons of different colours meet, they both simultaneously change colour to the third colour (e.g., if a grey and brown chameleon meet each other they both change to crimson). Is it possible that they will eventually all be the same colour?

Comment on solution 7

Looking for an invariant is a standard method of proof. The easiest invariants to spot in a real life situation are preservation of parity. This problem is a little more difficult to solve but I have included it here because I like it so much. The situation in which two dull coloured animals can both turn to crimson after touching is interesting. When you see the solution it is not so difficult. I would strongly encourage the reader to solve this without looking at the solution. However I do give it here.

In this case the numbers of chameleons of each colour at the start have remainders of 0, 1 and 2 when divided by 3.

Each 'meeting" maintains such a situation (not necessarily in any order) as two of these remainders must either be reduced by 1 (or increased by two) while the other must be increased by 2 (or reduced by 1).

Thus at least two colours are present at any stage, guaranteeing the possibility of obtaining all of the three colours in fact by future meetings.

Conclusions

The above problems do not necessarily connect to the Study document. I include them to emphasise cases where some accessible techniques have been used with secondary students to provide challenging situations. They are also a collection of problems which I obviously like very much also.

The Study has wide scope and has significant implication. This interaction between school syllabus and challenge has not been looked at on this scale before.

I encourage you to take an interest and read the web site at http://www.amt.edu.au/icmis16.html, where the discussion document and further information can be found.

References

Niven I. (1965). *Mathematics of Choice: How to Count Without Counting*. Mathematics Association of America.