

Mathematics ~ making waves



Proceedings of the Nineteenth Biennial Conference of
The Australian Association of Mathematics Teachers Inc.

Edited by
M. Goos & T. Spencer



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ISBN 1 875900 53 5

Published by

The Australian Association of Mathematics Teachers Inc.

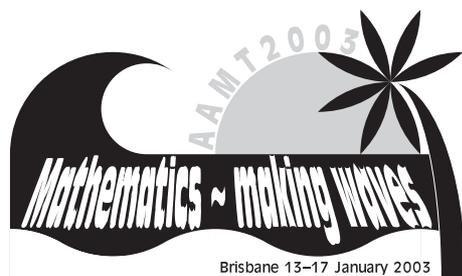
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Preface

The Nineteenth Biennial Conference of the Australian Association of Mathematics Teachers was held at The University of Queensland, Brisbane, from 13–17 January 2003. The conference attracted delegates from all Australian States and Territories as well as visitors from several overseas countries.

The Proceedings of the conference are published by AAMT and are available as PDF on CD-ROM. A copy of the Proceedings was provided to all conference delegates, and extra copies are available for purchase from the AAMT office.

Presentations at the conference were in the form of either papers or workshops; all papers were required to submit a formal written paper for inclusion in the Proceedings, while this was optional for workshop presenters. Some written papers were subject to a refereeing process overseen by the Chair of the Program Committee (Merrilyn Goos). AAMT members and people who had previously presented papers at AAMT conferences were invited to act as referees, and the responsibilities of this role were made clear to those who accepted.

Papers submitted for refereeing were subject to blind peer review by two referees. As far as possible, the expertise and interests of referees was taken into account when papers were allocated for review. Referees were asked to assess the significance of the paper, its appropriateness for this conference, the quality of the conceptual or practical ideas it presented, and the quality of writing. They were then asked to judge the paper as either ‘acceptable’ or ‘not acceptable’ and to provide brief comments justifying their decision. In these Proceedings, refereed papers denoted ‘acceptable’ are identified by the symbol (*) and a footnote indicating that the paper has been subject to peer review.

Papers

The effectiveness of the graphics calculator in solving equations and functions skills for Omani prospective teachers

Reda Abu-Elwan

Sultan Qaboos University, Muscat, Oman

A significant amount of research has been conducted into the effectiveness of the graphics calculator as a tool for instruction and learning within pre-calculus and calculus courses, specifically in the study of functions and graphing. This study aimed to investigate the effectiveness of graphics calculator in solving equations and functions for Omani prospective teachers, as a new technology tool used with Omani students. The main objective of the study was to determine the graphic calculators skills needed to teach that topic in an Algebra course. Twenty-five students from the Mathematics / Computer department in the College of Education were involved in the experimental study, using the Casio cfx9850+. Results of the study showed that a graphic calculator requires new skills to be used effectively (as numerical, graphical, symbolic and translations). Graphics calculators are an effective tool to enhance equation solving and functions skills for the experimental group of the study. Participants attitudes were significantly positive towards the graphics calculator.

Introduction

There is a growing consensus among mathematics educators that school mathematics should be portrayed as a useful and vibrant subject to be explored and understood. This contrasts with the traditional view of mathematics as an immutable and absolute set of rules and procedures to be mastered. This consensus implies changes in both what mathematics is learned and in how it is learned.

The reform movement in mathematics education emphasises not only a shift in what is learned, but also in how it is learned. The NCTM recommends that scientific calculators with graphing capabilities be available to all (secondary) students at all times (NCTM, 1989, p. 124). The *Standards and Principles* (NCTM, 2000) stresses the importance of appropriate technology integration in the curriculum.

Teachers have the responsibility to make appropriate choices about the use of calculators in particular contexts, on the basis of how well the tool will help or hinder

the intended mathematics development. Technology is an important part of our world, and schools must affirm the appropriate use of calculators at all levels as a legitimate and important tool for learning and doing mathematics.

The graphics calculator has specifically developed as a tool to assist the teaching and learning mathematics. Graphing calculators can empower students to use their ability to visualise, as well as developing mathematical skills and concepts, to work mathematically. Students can use graphing calculators to create graphing representations that will enable them to give geometric interpretations to algebraic concepts. Also, they can use it to see reasons for some algebraic procedures, to confirm algebraic solutions, to check algebraic manipulations, to develop deeper understanding about the content of algebra, and to build understanding about many topics in connection with algebra. (Dunham & Dick, 1994).

In Oman, little attention has been paid to using graphics calculators in secondary education and at university level, all kinds of scientific calculators have been widely used. One of the main purposes of the new Omani mathematics curriculum at secondary level is to engage students in mathematical modelling, it is therefore essential that students have some understanding of the graphing approach taken by graphics calculators and realisation that the picture obtained may not be exact.

Objectives

The present study aims to:

1. examine the availability of using graphics calculators in solving equations and functions by Omani prospective teachers;
2. determine teaching skills needed for prospective teachers to teach algebra using graphics calculators;
3. determine prospective teachers attitudes towards graphics calculators.

Research questions

1. How will graphics calculators affect the achievement of solving equations and functions by Omani prospective teachers?
2. What are the teaching skills needed in teaching algebra using graphics calculator?
3. What are the attitudes of Omani prospective teachers toward graphics calculator?

Theoretical framework

It is known from previous reviews of research into graphics calculator usage in mathematics education (Penglase & Arnold, 1996) that graphics calculators can be used as a great technological tool to assist both students and teachers to solve problems related to algebra (Hackett & Kissane, 1993). They are also used as a part of teaching methods, when researchers have integrated it with various teaching approaches.

Emese (1993) investigated the effects of graphics calculator use when integrated with a guided discovery style of teaching (a lecture/discussion instructional technique) within

a university differential calculus course. Barrett and Goebel (1990) concluded that there are two areas of instruction in which graphing calculators are likely to have the most immediate impact for solving equations and analysing functions, in addition to these areas, graphics calculators can enhance instruction in the topic of data analysis. This paper focusses on the effectiveness of using graphics calculators in solving equations and functions as an effective tool to enhance prospective teachers achievement, and teaching skills needed for teaching algebra using graphics calculators. In this study , I did try to investigate the attitudes of the experimental group towards graphics calculators in teaching because this was the first time they had each used a graphics calculator.

Solving equations and functions

It seems that a graphics calculators are a very useful tool in making mathematics lessons more meaningful and interesting to students, and makes graphing of difficult functions easier and faster, which can be time consuming and inaccurate when done manually. It allows investigation of functions through tables, graphs and equations in ways that were not possible before their proliferation (Beckmann & Thompson, 1999). Graphing calculators allow the focus to be understanding and setting up and interpreting results (Dick, 1992).

An important area of investigation involves using graphical methods to solve problems containing a system of linear equations that are traditionally solved algebraically. A problem contains a system of linear or non-linear equations might be solved by tracing then zooming in on the intersection point(s) of the graphs.

Previous research showed that graphics calculators are able to make stronger links between graphic and algebraic representations of functions (Ruthven, 1990). Rich (1990) elaborated in her study that students in two high school pre-calculus classes had a better understanding of algebraic problems when taught using graphics calculators.

In the Omani secondary mathematics curriculum, the only methods of solving equations are algebraically and graphically by hand: graphics calculators are not used. This was the first time graphics calculator had been used to solve equations by Omani prospective teachers, so it was important to investigate their achievement.

Skills for teaching algebra using a graphics calculator

Skills needed for teachers who teach with a graphics calculator are still uncertain, and studies have focussed upon the effects of different teaching styles on graphics calculator usage and the effectiveness of graphics calculators when they are integrated with various teaching approaches (Penglase & Arnold, 1996). Strait (1993) compared the effectiveness of two teaching strategies that included use of graphics calculators, namely deductive and inductive approaches, which were used in teaching a unit on functions and analytic geometry as part of a college algebra course. Results of a pre-and post-test were analysed to compare students procedural skills found there was no significant difference between the two groups of students in procedural skill development. Rich (1991) noted that teachers who taught a precalculus college algebra

course with the aid of graphics calculators, asked more higher order questions, used examples differently, and stressed the importance of graphs and approximation in problem-solving to a greater extent than teachers who taught the same course without the use of graph calculators.

This paper focusses on teaching skills needed for teaching algebra using graphics calculators.

Attitudes toward graphics calculators

Students who use graphics calculators have a positive attitude towards the graphics calculator, but these attitudes vary when using the calculator to solve functions or working in statistics. In a survey (Boers & Jones, 1992) of students' attitudes towards graphics calculators in University of Technology, Australia, it was found that students attitudes have been very positive toward the use of graphics calculator, and students tended to feel that the introduction of this tool had resulted in them adopting mathematical practices which aided their learning.

On the other hand, it has been observed that while many junior mathematics teachers are positive about graphics calculators, attitudes of experienced teachers are much more varied (Routitsky & Tobin, 2001). Teachers were asked to respond to several statements about the use of graphics calculators for teaching and learning, even if they did not use graphics calculators, more than 50% of teachers who responded to the questionnaire, believed that graphics calculators were useful in teaching. As mentioned previously, it was a first time our mathematics students in the College of Education had used graphics calculator as a tool to achieve skills of solving equations and functions, and to teach these topics using graphics calculators; so I tried to investigate their attitudes toward graphics calculator.

Methodology

Participants

The participants of this study were 25 student teachers; all of them were female and enrolled in the *Teaching Practice II* program during fall term 2002; they were from mathematics/computer department in College of Education, Sultan Qaboos university. In this study, all students represent the experimental group.

Instruments

Three instruments have been developed and used to test hypotheses:

1. *An achievement test on solving equations and functions* has been developed to determine the effectiveness of using graphics calculator to enhance student teachers' performance. The test consists of 8 questions, all of it has been chosen from three main resources: Kissane (1997; 2000) and Internet resources.

Questions were chosen to measure student teachers' skills in solving equations and functions using their graphics calculator. Five referees were asked to give

their opinions regarding the validity of the test, and changes were made based on their suggestions. Reliability of the test was established using another group of 20 mathematics student teachers in grade three at the College of Education; a reliability coefficient mean value of 0.71 was secured.

2. *Observation sheet*: This was developed to measure teaching skills of student teachers when they use graphics calculators. It concentrates on teaching skills related to using graphics calculators for solving equations and functions. A group of mathematics supervisors from the Ministry of Education were asked to give their opinions about expected student-teachers behaviour in teaching mathematics using a graphics calculator; opinions were collected and the items were developed. The observation sheet consists of 12 items that classified under three main tasks:

- (a) Introduction teaching skills (3 items: 1, 2, and 3)
- (b) Teaching methods skills (5 items: 4, 5, 6, 7, and 8)
- (c) Evaluating skills (4 items: 9, 10, 11, and 12)

Responses were collected on a scale from: very little (1), little (2), high (3), very high (4), not appear (0). Reliability coefficient for mean value of 0.73 was secured.

3. *Attitudes toward graphics calculators scale (ATGCS)*:

The objective of (ATGCS) was to measure student teachers' attitudes toward graphics calculators in teaching and learning. Student teachers used graphics calculators to learn the topic of solving equations and functions graphically and they used it as a tool of teaching the same topic in microteaching situations. (ATGCS) has been developed based on previous scales (e.g. Routitsky & Tobin, 2002) and for the purposes of the study, (ATGCS) has been presented to referees from Psychology and Curriculum departments in College of Education. Responses were collected on a scale from: somewhat agree (4), agree (3), disagree (2), somewhat disagree (1), I don't know (0). The sheet reliability was then established using the inter-rater agreement coefficient, based on the data gathered for this purpose a reliability coefficient of 0.70 was obtained.

Design of the experiment

Part 1

A pre-test was presented to the experimental group of the study before teaching the developed graphics calculator activities; student teachers were asked to solve it by only using their graphics calculator, then all participants were involved in learning activities related to solving equations and functions. All worksheets have been developed from three resources: Kissane (1997), Kissane & Harradine (2000) and The University of Melbourne (1997). Main topics in the teaching activities:

- introduction to graphics calculator (basic operations, graphing, etc),
- linear equations,

- quadratic equations,
- tables of values and graphs,
- handling functions,
- drawing a graph of a function,
- tangent to a curve.

Each student was supplied with a graphics calculator (Casio cfx-9850GB+), by the university for the purpose of the study. Students did all required activities with their graphics calculator as 'self learning' with guidance from the researcher. The program took two hours on three days of each week, and lasted for three weeks: the total time was 18 hours, and in addition to that students were allowed to take the calculators with them in order to do further practice and training. After these learning activities, an achievement test on solving equations and functions, the same as the pre-test, was presented for students as a post-test, and they were asked to solve it by only using their graphics calculator. Data has collected and analysed.

Part 2

Participants were asked to prepare a mathematics lessons for grade one of Omani secondary school where the 'solving equations' topic was included. Ten participants then presented these lessons to their classmates in microteaching situations; graphics calculators were used in the teaching process, and all lessons were observed by researcher using the designed 'observation sheet'. It took five days, with two lessons presented in two hours each day. The *Attitudes Toward Graphics Calculators Scale* (ATGCS) was applied at the end of the microteaching practices.

Data analysis

Data analysis was completed through use of statistical package SPSS (v10.1.0) for:

1. pre-post test on 'Solving Equations and Functions Skills';
2. observation sheet for teaching skills needed when using the graphics calculator;
3. scale of attitudes towards the graphics calculator.

Results

Research question #1

The study was conducted with one experimental group consists of twenty-five student-teachers. An achievement test was used as the measurement in an experimental pre-test/post-test design. Paired-sample t-test was used as a measure of comparison between the mean scores of the pre-test and post-test.

Table 1. Means and standard deviations of scores on the achievement test.

	N	Mean	Std Deviation	T-value	P<
Pre-test	25	15.72	3.94	-72.65	.000
Post-test	25	31.68	4.04		

Maximum score is 8questions X 5 marks = 40 marks

It is evident from Table 1 that the level performance of 'solving equations and functions skills' using graphics calculators has significantly improved over the time of the study. The two-tail probability of $p < .000$ indicates that these increases are very significant.

Research question #2

Ten participants were observed teaching lessons in solving equations for grade one at secondary school; use of the graphics calculator was incorporated into these lessons, then they were presented in microteaching situations in class. The number of student teachers was limited due to time allowed to use lab, the number of lessons from the textbook; I believe that student teachers who made presentations displayed many different ways of using the graphics calculator in teaching. Results of the observation sheet are shown in Table 2.

Table 2. Means and standard deviations of teaching skills.

N	Content of items (Teaching skills)	Mean	Std Deviations
1	Introducing requirement of lesson by graphics calculator	2.1	.994
2	Introducing basic functions of calculator keys	3.5	.527
3	Be sure that student using keys properly	1.6	.699
4	Teaching activities are very clear in lessons	3.7	.483
5	Guiding students individually during work in equations	3.4	.516
6	Give chances for students to cooperate with each others	1.7	.483
7	Give a brief conclusions about class work	3.5	.527
8	Control of time and teaching activities	3.7	.483
9	Ask each student about his performance and solve their difficulty in calculator	3.3	.483
10	Verifying and correct students' performances	3.6	.516
11	Present assessment questions on graphics calculator uses	1.5	.527
12	Introduce an idea about next lesson	1.6	.516

Research question #3

Student teachers were asked to respond to (ATGCS) after they finished their teaching presentations. Responses were collected — on a scale from 0 ‘I don’t know’ to 4 ‘Strongly agree’ — in two sections: first to measure attitudes toward graphics calculator use for learning algebra (questions 1–8); second to measure attitudes toward graphics calculator use for teaching algebra (questions 9–15). Analysis of responses showed that positive answers ‘st. agree’ and ‘agree’ were more common than ‘st. disagree’ and ‘disagree’. This is shown in Table 3).

Table 3. Means, Percent and Std Deviations for (ATGCS).

Item N	Mean	Std Div	Item N	Mean	Std Div
1	3.64	.76	9	3.4	.87
2	3.28	.89	10	3.32	.9
3	2.04	.68	11	3.32	.9
4	3.12	1.13	12	3.36	1.04
5	1.2	.87	13	3.08	1.19
6	2.88	1.27	14	3.2	1.26
7	1.28	.84	15	3.48	.87
8	1.76	.66			

Discussion

Results from research question #1 showed that the pre-test mean score was 15.72 and standard deviation was 3.94; while the post-test mean score was 31.68 and standard deviation was 4.04, t value was -72.65 and this value is statistically significant at level $p < 0.000$. This indicates that the difference was due to the use of the graphics calculator in learning how to solve equations. It seems that the mean score of the pre-test was not high because of weak knowledge in how to use the graphics calculator, but at least student teachers could use calculator functions and they were allowed suitable time to do the test. In the post-test, student teachers’ performance was in high scores because of the work done in solving equations using graphics calculators. It was noticed that attention needs to be paid to the precision limitations of the machine.

Results of research question #2 and Table 2 showed that responses of student teachers were high for the categories (very high & high) for items: 4(3.7, .483), 8(3.7, .483), 10(3.6, .516), 7(3.5, .527), 2(3.5, .527), 5(3.4, .516) and 9(3.3, .483). Range of means was (3.7–3.3). Performances of student teachers were significantly satisfied on skills 4, 8, 10, 7, 2, and 9 listed on Table 2. Other teaching skills have had a lower appearance for categories (little and very little) on items: 1(2.1, .994), 6(1.7, .483), 12(1.6, .516), 3(1.6, .699) and 11(1.5, .527). Range of means was (2.1–1.5), indicating that other skills 1, 6, 12, 3 and 11 were not present in the student teachers’ performances. Results of research question #3 as it shown in Table 3, showed that responses from student teachers were high on the categories ‘St. Agree and Agree’ for items: 1(3.64, .76), 2(3.28, .89), 4(3.12, 1.13), 9(3.4, .87), 10(3.32, .9), 11(3.32, .9), 12(3.36, 1.04), 13(3.08,

1.19), 14 (3.2, 1.26) and 15(3.48, .87) Range of means was (3.64–3.08), and responses showed that there is a positive attitude toward the use of graphics calculators in learning and teaching algebra. Participants were very interested in using graphics calculators because it was their first experience working with them. This result is consistent with other previous findings (e.g. Khairiree, 2001). Other responses for items: 3(2.04, .68), 5(1.2, .87), 6(2.88, 1.27), 7(1.28, .84) and 8(1.76, .66) showed that student teachers would like more experience with graphics calculators and respond positively to these questions which focussed on their interest in finding more information about graphics calculators and their use.

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Acknowledgements

The support from Sultan Qaboos University in Oman for support given to this research under grant # EDU/03/00 is greatly appreciated.

Teaching mathematics and the Web: A task-object approach*

Alan Barnes & Esther Yook-Kin Loong

University of South Australia

Some of the key learning areas such as SOSE and English have had long standing pedagogies that have taken educational advantage of the increasing resources on the World Wide Web. This paper explores the possibilities for similar congruencies between traditional mathematics teacher practices and mathematics resources on the Web. It looks at a range of existing teacher practices and analyses them in relation to a typology of mathematically relevant Web resources. A 'task-object' methodology is proposed where specific teacher practices or student tasks in existing curricula (and textbooks) are replaced or enhanced by the use of Web objects of relevant mathematics functionality. Examples are given and possible advantages for student learning discussed. The paper will describe a math learning object database project, which aims to provide 'curriculum- or task-driven' access to web objects.

Key learning areas, learning technologies and the Web

While there are many case studies of use of the World Wide Web in teaching there is little data on the broader picture of use of the Web across subjects and teacher specialisations in schools. The Australian *Real Time* study (Meredyth et al., 1999) did examine the use of information technologies across key learning areas in a survey of some 1258 teachers in 1998. The key learning areas (KLAs) were ranked in terms of information technology usage from the highest as: English, Society and the Environment, Science, Mathematics, and Technology and Enterprise. Other KLAs such as Languages other than English (LOTE), the Arts, and Physical and Health Education made little use of computers in the classroom. In 1998, Mathematics rated highest among KLAs for the use of educational software/games but lower for informational and creative uses and much lower for use in communication. Overall, Mathematics was rated fourth of the eight KLAs for information technology use.

In Australia since the *Real Time* study there has been a surge in expenditure on school information technology and a dramatic increase in access to the Internet in schools.

* This paper has been subject to peer review.

More recent Australian data from the South Australian ‘Discovery Schools’ (Filsell & Barnes, 2002; Barnes & DETE, 2002) shows the increased uptake of Internet-related technologies.

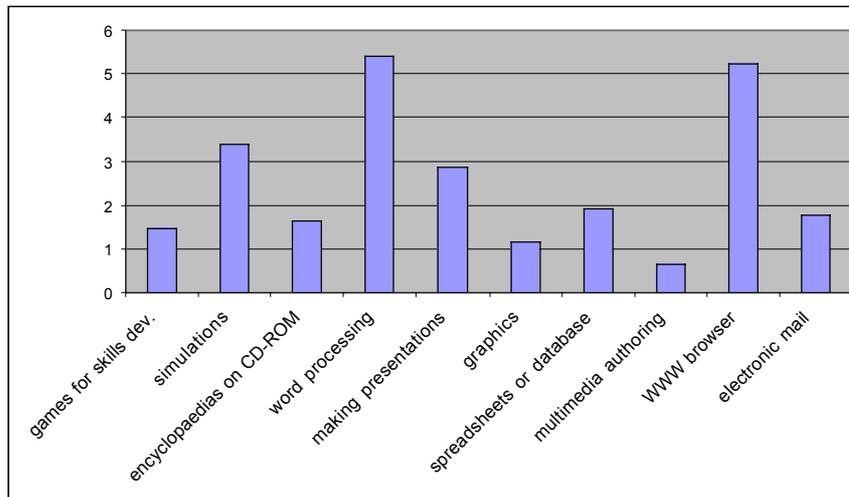


Figure 1. Mean usage of types of learning technologies in 2001 in the two discovery high schools, for 118 staff. The vertical axis shows the number of lessons in a term.

The data in Figure 1 shows the planned usage in lessons per term for learning technologies in two high schools in the discovery program. The data comes from a detailed staff survey, carried out by one of the authors in conjunction with the South Australian Department of Education, Training and Employment (DETE) over the three years of the discovery program (Barnes & DETE, 2002). The survey asked staff a number of questions in relation to their use of the Web, their priorities and confidence. Staff also indicated key learning areas taught. The following charts show individual responses to these variables for staff who indicate they teach mathematics and/or other learning areas (LAs). The data does not relate to class teaching in a specific learning area – a more complex survey task beyond the scope of the initial research. It is about mathematics teachers rather than mathematics teaching. Many teachers teach more than one learning area and while the data is useful for ranking purposes, it should be used with care when making ratio comparisons between different learning area indices.

Staff who report teaching mathematics are ranked eighth out of the ten learning areas above in their adoption of learning technologies in their classes. As in the *Real Time* study, teachers of English and SOSE are the biggest users of learning technologies but mathematics and science teachers have also fallen behind Arts and LOTE in this particular survey. Comparison of learning technologies across learning areas suggests teachers of mathematics are above mean only for ‘making presentations’ or ‘spreadsheets and database’. Mathematics teachers rank highest in terms of spreadsheet and database usage followed by computing and science.

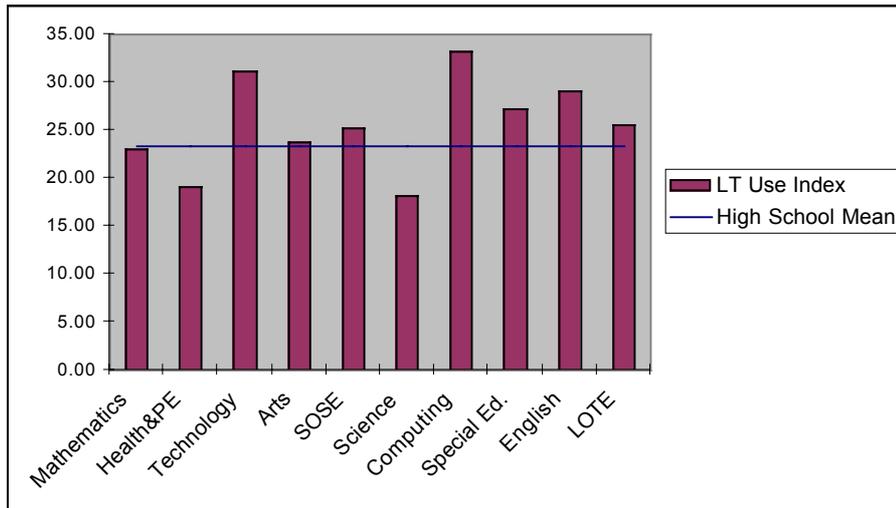


Figure 2. Reportage of lessons using learning technologies (LT) per term of staff claiming teaching in particular learning areas. The LT Use index is an aggregate of the learning area uses in Figure 1 above.

In relation to the Internet, Figure 3 shows that for the World Wide Web, mathematics teachers lag behind teachers of LOTE, Special Education, English, Computing, SOSE and Technology. Mathematics teachers provide less than the mean number of lessons using the World Wide Web. Use of e-mail shows a similar picture.

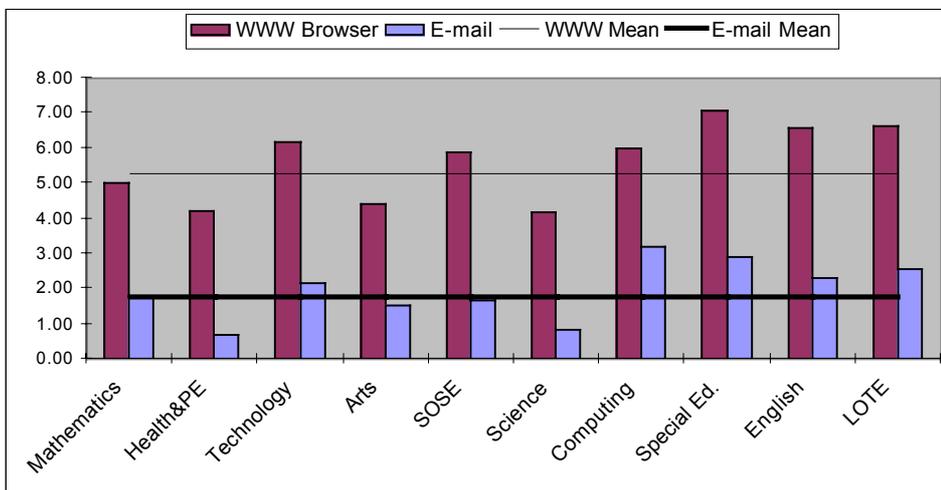


Figure 3. Reportage of lessons involving the World Wide Web and e-mail by teachers citing various learning areas.

In another series of questions teachers were asked to indicate how often they use technologies for preparation, or reporting. The scale ranged from never to always with 'never = 0', 'rarely = 1', 'occasionally = 2', 'frequently = 3', 'always = 4'. Figure 4 below

shows two Internet related activities about getting information from the Internet (usually the Web) for use in lessons and about posting student work, and resources on the web.

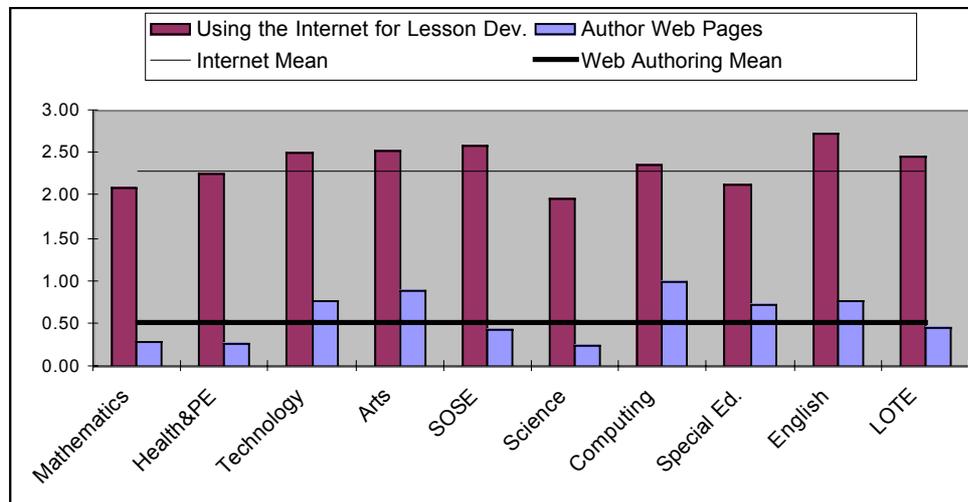


Figure 4. Reportage of the frequency of two teaching preparation tasks by teachers

The data suggests that mathematics teachers are less inclined to use the World Wide Web in their preparation and hardly ever use the Web as an authoring environment.

The data does not suggest teachers of mathematics do not use the World Wide Web. Indeed web usage increases for all categories of teachers across the three years of the discovery program. Mathematics teachers tend to take up the Web less than their peers in many other learning areas (science excepted). Moreover, mathematics teachers' priorities are consistent with this lower than mean usage. Teachers were also asked to indicate their valuation for a range of equipment options. Generally mathematics teachers thought both a computer with Internet connection and WWW access in class would be valuable for their own use but not as strongly as teachers citing any other learning area.

A final set of questions in the staff survey asked about confidence in various areas. Mathematics teachers in fact showed higher than average confidence with search engines and with Web authoring for 2001. Further detail on the use of learning technologies by teachers in the discovery schools will be available in a forthcoming paper.

A picture emerges of mathematics teachers confident in their use of the World Wide Web but less likely than many other learning areas to utilise the Web in either direct teaching or in preparation. None of the above in any way suggests that such teachers are not effective in their current teaching practices, however they do seem less likely to take up the new opportunities offered by the World Wide Web compared to many of their peers.

What is at issue here is not mere usage of the Web in teaching and learning, but the adoption of effective teaching and learning practices that make use of the technology. Some of the learning areas above have had long standing pedagogies that can work effectively with Web materials. The well-developed pedagogy of ‘resource’-based learning for example was widely used in SOSE and English prior to the web, but the advent of the Web made accessible a huge array of useful resources and saw a considerable expansion in this approach to learning.

In the following we seek just such congruence between traditional mathematics teaching practices and Web materials. To do this, we encourage a different view of Web materials and examine their relevance to mathematics teaching.

Teaching strategies and Web objects

A typology of mathematical Web objects

There is on the World Wide Web an array of ‘materials’ of varying relevance to mathematics education. These materials might be websites, collections of web pages or web objects within the pages. Some materials may have embedded within them sophisticated mathematical algorithms that govern their interaction with the users. In an attempt to bring some order to thinking about these materials, Loong (2001) chose to consider the materials as being composed of ‘learning objects’ and made an initial attempt to categorise them in relation to their functionality and characteristics. The approach further extends that of the Learning Object Model promoted by the Learning Standards Committee of the IEEE (Suthers, 1999) and that of the Learning Federation Australia (2002) in its role in developing Australian educational content for the web.

This notion of ‘learning objects’ is crucial in setting a new paradigm of use for mathematics teachers. Whereas in many learning areas such as SOSE, Geography and Language Studies, web materials are often considered and pointed to in terms of whole sites or collections of web pages, we are proposing a ‘learning object’ approach for the teaching of mathematics. This approach seeks to identify objects of mathematical relevance and signifies their value to specific teacher or student tasks.

Web pages can be considered as composed of objects that can be described as either ‘resources’ or ‘communication’. Consistent with the Learning Object Model, resource groups have been categorised as either ‘interactive’ or ‘non-interactive’.

Interactive resources were categorised as having functions that give feedback to the user or support explorations. Interactive objects engage with learners or teachers in some way and respond based on their programming or functionality. Learning objects commonly found with feedback functionalities include objects like exercises, games, and calculators. In certain instances, the feedback given is a closed one, for example, ‘Incorrect. Please try again’, or it could provide a link to an explanation. Exploratory objects found on the Web are those with capabilities that support student manipulation through changing parameters or settings, or sometimes following various commands and rules to construct mathematical representations. Objects categorised as *Non-animated investigations* are those which when manipulated respond directly with a change in representation. *Animated investigations* are objects that respond to user

changes by presenting an animation, for example a rotation, flyby or locus. Interactive objects all contain some mathematical functionality; some objects exhibit a simple numeric function; others may exhibit considerable symbolic manipulation or mathematical computation. Such functionality is sometimes generated by the user's computer through Javascript or Java applets, or may be generated by a server with specific mathematical software.

Non-interactive resources have been grouped as materials that are 'text-rich' or 'graphic-rich'. Text-rich materials, as the name implies, are materials that are rich in information and are predominantly in text form (with some in symbolic form). These materials have been grouped into the respective learning objects:

- research articles;
- expositions of concepts and mathematical ideas;
- teaching documents such as lesson plans, worksheets or databases, etc.;
- material with authentic life applications such as narrative text about banks or businesses;
- mathematics related jokes; and
- non-interactive pages of exercises and problems sets.

The other group of non-interactive materials are those that are presented in the graphic form. This category of materials can further be grouped as *general images* with mathematical relevance, for example, a picture of a geodesic dome, mathematical *diagrams* such as 2 or 3-dimensional shapes, graphs, charts or tables, or images of mathematics-related *cartoons*.

The number of people who use the Internet for electronic communication has grown considerably. Besides such direct communication with one another, the communication opportunities afforded by the Internet includes those on the World Wide Web where the user can participate in online discussions or threaded discussions made available from websites. There are numerous teacher forums and open forums that are educational, as well as those set up specifically for students. Since the mid-1990s several question and answer panels that specifically cater to mathematics have been established. Panel members are drawn from school mathematics teachers, mathematics educators and university students. Where responses have been overwhelming, the questions and answers have been archived but the facilities for such communication still exists.

This paper looks at this typology from a teacher practice perspective. Careful study of the characteristics and functionalities found in the learning objects lead to a consideration of the types of teaching strategies that can be used with them. Four main categories of teaching strategies have thus far been identified. They are: expositive strategies, active strategies, collaborative strategies and consolidation strategies. This mapping of the typology to teacher practice is not meant to be prescriptive but merely to demonstrate the possibilities for mathematics teacher usage of the Web. The mapping is presented in a graphical form in Figure 5.

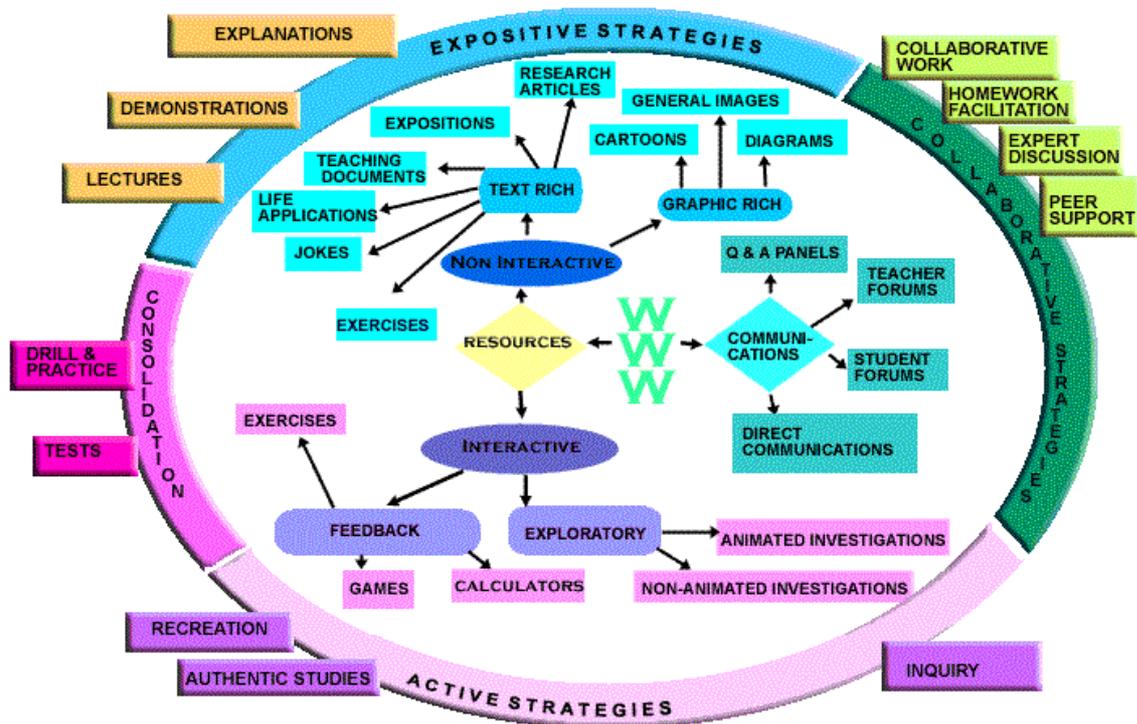


Figure 5. A typology of Web objects and the teacher practices they can facilitate.

Web-based teaching strategies

Mathematics has long been thought of as a domain for abstract or logical thinkers. As such, the teaching of mathematics has been dominated by practices that attempt to build on those abstract capabilities. However, as Gardner (1983) argues, there are other types of intelligence that may be involved in the learning process. In considering the World Wide Web as a resource for mathematics teaching and learning, one should consider how best to exploit the functionalities found on the Web so that it caters for the diverse learning differences found in our students. The following describes some groups of strategies that could be applied to a Web-based environment.

Expositive strategies

Expositive strategies such as lectures, demonstrations or explanations can be used with learning objects that are text-rich or graphic-rich. Materials from such learning objects can be used to enhance teacher preparations and presentations. Traditionally, this approach is used to transmit information from the teacher to the student, and if not carefully planned can be boring and poorly presented. This approach can be more effective if suitable web objects are used to stimulate or motivate students as well as to demonstrate key concepts. Interesting learning objects could be used at the beginning of lessons as attention grabbers and to establish motivation to follow through to the body of the lesson. Although in Figure 5 where expositive strategies are used with non-

interactive text-rich or graphic-rich material, a teacher could equally use interactive explorations as a springboard for classroom exposition. Many of the dynamic and interactive applets can be used to demonstrate key ideas or as set induction. This teacher-centred approach can be turned into one that is student-centred when students are directed to do a range of tasks with text-rich explanations or concepts found on the Web. Although expositive strategies like these are better disposed to key learning areas like SOSE or Language Studies where text plays a vital role, their value in students' mathematics project work or search for historical progress of mathematical ideas should not be discounted.

Active strategies

Web objects that are high in their interactivity lend themselves well to being used in a constructivist way. The dynamic and interactive nature of these objects allows students to manipulate and change parameters, and view the resulting effect. Active (constructivist) strategies that utilise dynamic learning objects for explorations allow abstract concepts to be represented visually and interactively. These have the potential to cognitively and visually engage the learner in a way that is not possible through static symbolic forms. There are increasing numbers of such learning objects on the Web that mathematics teachers can utilise for their teaching. Inquiry methods of teaching and learning can be used with such exploratory learning objects. The teacher sets the problem or activity and students explore the situation; the teacher and students can then discuss the situation and draw their conclusions. Under such circumstances, the students sharpen their observation and inquiry skills and begin making conjectures and postulations rather than being passive receivers of knowledge. The teacher's role is more of a guide and questioner rather than a provider of knowledge. While broader inquiry activities are becoming an increasing feature of mathematics curriculums such as SACSAP (Department of Education, 2001), the object-based explorations above have value in focussed concept development. It is a characteristic of such broad investigations that they are lengthy and require considerable teaching resources in their implementation and evaluation. However in the crowded curriculum, the use of exploratory web objects that are specifically focussed around mathematics concepts can be less consuming of time and resources.

Feedback systems such as those found in some Web mathematics games enable students to learn mathematics in a recreational way, in addition to reinforcing certain mathematics concepts or capabilities, and can replace some dreary repetitive mathematics exercise in the textbook. Investigations with calculators such as those found in some bank websites or business websites can be used to replace exercises that hold little relevance to everyday life situations. By establishing certain scenarios, and using information found on the Web, mathematical modelling problems can be set. Instead of using exercises as a practice for recently learned procedures, students can construct a mathematical model based on a real life situation, solve the model for mathematical solutions, interpret these solutions in the context of the real situation, and refine the model to produce better solutions (Herrington, Sparrow et al., 1997)

Consolidation strategies

There are both interactive as well as non-interactive exercises on the Web. The non-interactive exercises are not different from those found in textbooks, and their value may be one of complementing textbook problems through increased difficulty levels or variety. On the other hand, interactive exercises have great value. Interactive feedback systems found in Web exercises have the potential to encourage independent self-monitoring and evaluation among students. Besides the drill and practice afforded by such objects, there is also motivational value in giving instant feedback on the student's progress. In some systems, hints are given and the student redirected to the section of the module that explains the concept.

Collaborative strategies

Collaborative work on the Web has been shown to increase students' awareness of the many and varied solutions to open-ended problems: because of the influence of other students' notes, by becoming aware of the general formula others use, and by having access to many mathematical views and conceptions (Nagai, Okabe et al. 2000). Peer support is clearly one of the advantages of Web communications. Online mathematics clubs on the Web allow individuals who share similar interests to feel they belong to a community instead of feeling isolated (Way & Beardon, 2001). The teacher can encourage the use of communications with peers and experts to facilitate homework: the excitement and fulfilment from getting into a discussion with experts from anywhere in the world can be an uplifting experience.

The task-web object approach

In mathematics, curriculum is usually structured and sequenced, and understanding is developed through mastery of first elementary and then more complex topics. Mathematics teachers will present expositions and methods but a large amount of student time is involved in the active doing of tasks in examining proofs and methods, and carrying out exercises from worksheets or textbooks. Web objects could be utilised in many of these tasks. Many of these web-based tasks could be carried out as homework.

The task-web object approach would see some of these tasks enhanced or substituted by the appropriate use of web objects.

Table 1: Possible task replacements or enhancements with appropriate web objects

Strategies	Replace/ Enhance	With
Consolidation	Textbook and worksheet exercises	Interactive games http://www.aplusmath.com/cgi-bin/games/geomatho Interactive exercise with feedback and clues http://www.mathcounts.org/GoFigure/Main.taf?function=Start&NewQuestion=True&LevelID=101&CategoryID=100
	Teacher Evaluation	Student self evaluation through web tests http://www.bbc.co.uk/apps/ift/schools/gigaquiz?infile=angles&path=gcsebitesize/maths/angles
Active	Explore a graph manually	Dynamic exploratory investigations http://www.univie.ac.at/future.media/moe/galerie/fun2/fun2.html#sincostan
	'Real-Life' Problems in textbook	Authentic case studies e.g. bank calculators, weather studies, real databases https://services1.anz.com/nola/application/hl/help/Information.asp
Expositive	A flat colourful diagram	Animated 3-dimensional diagrams http://www-sfb288.math.tu-berlin.de/vgp/content/curve/PaSurfCurve.html Dynamic investigations
	Expositions on concepts	Student initiated search for information on the Web http://library.thinkquest.org/3288/fractals.html Inquiry led investigations
Collaborative	Giving answer to a difficult homework question	Ask an expert, student forum http://www.nrich.maths.org/discus/messages/26/26.html?1035306537

The effectiveness of the task-web object approach will rely on teachers having direct access to objects that match particular task needs. To that end the authors have set up a prototype database that demonstrates how access to relevant web objects can be supported. This database can be accessed via the following web site <http://www.education.unisa.edu.au/elearn/w3mathsed/> under 'A search facility' in the 'Resources' webpage. In addition to 'key words' fields, other relevant fields that can be used to facilitate a search in the database include: branch of mathematics, key ideas, year level, skill level, interactivity type, object type, and targeted education system.

Conclusion

For mathematics education to effectively use the Web, appropriate pedagogies will have to be used. The task-web object approach is an attempt to suggest a feasible practice that can be used by mathematics teachers. By capitalising on students' enthusiasm for things on the Web, mathematics teachers could enhance student interest in mathematics through using the replacements or enhancement strategies suggested.

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Searching for a complete graph: Approaches using a graphics calculator*

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Expertise in finding a global view is essential for senior secondary mathematics students studying functions. An exploratory study investigated student approaches using a graphing calculator in a non-routine situation. The study found that the following had the potential to facilitate the sketching of a complete graph: (a) graphing calculator features dedicated to transforming the viewing window, (b) directly altering the window settings, (c) graphing calculator features enabling key features of the function to be identified, (d) the linking of mathematical and graphing calculator knowledge, (e) planning, and (f) use of the table facility.

Change in teaching and learning of functions in a graphing calculator environment

Functions have an important role in mathematics (van der Kooij, 2001; Zaslavsky, 1997). Graphs portray an essential aspect of functions providing insight additional to the algebraic representation. The ease of viewing multiple representations of functions in a graphing calculator learning environment, the open-ended nature of the tool, and the access to an increasing number of functions it provides could change the teaching and learning of functions. In addition, 'the global nature of the graphical image means that information can be extracted quickly and easily' (Arnold, 1998, p. 182). Use of a different tool will not necessarily improve student learning, even where teaching changes simultaneously to facilitate its use (van der Kooij, 2001). This study responds to Arnold's entreaty that 'if we are to learn to use these tools effectively, it becomes vitally important that we study the ways in which individuals make use of them within mathematical learning situations' (1998, p. 174). Even though hand-held technology has progressed since Arnold wrote these sentiments, to include computer algebra systems (CAS), Zbiek (2001) points out that 'secondary mathematics teachers... may find the lure of graphing capabilities is more compelling than the symbolic manipulation features of a CAS' (p. 3). For this reason, it is still imperative that

* This paper has been subject to peer review.

research continues to be conducted into how students use this technology in their learning.

The study

This exploratory study examined the approaches experienced student users of graphing calculators employed while working in pairs and attempting to solve a problem task using a graphing calculator. The task (adapted from Binder, 1995) involved students producing a sketch of the complete graph of a cubic function (see Figure 1). The task was selected because no part of the function is visible in the standard viewing window of the calculator used in the study, and because of the potential for the function to provide unexpected views as students search for a global view with a graphing calculator. These aspects ensured that the task was non-routine for the students.

Work cooperatively, using the graphing calculator, to sketch completely the graph of $y = x^3 - 19x^2 - 1992x - 92$.
Show all important features.

Figure 1. The problem task.

Participants

The five pairs of students who participated were in their last two years of secondary schooling and were studying the Victorian Certificate of Education [VCE] two year subject, *Mathematical Methods*, at an inner city state secondary college. Two of the pairs were in their final year of study of this subject while the others were in their first.

The researcher taught the Year 11 students, while the Year 12 students were taught by a colleague. These two teachers worked closely together and each had similar teaching styles. They both used a variety of methods emphasising understanding through exploration, discussion and collaboration.

With one exception, all participating students owned their own graphing calculator; the remaining student rented one from school.

Data collection

Data were collected in the form of audio recordings of the dialogue of students while undertaking the task, videotapes of the calculator output, written scripts, and observational notes. One Texas Instruments TI-83 graphing calculator, initially set to the standard window, was used by each pair. A *View Screen* and overhead projector allowed the projection of the calculator screen to be videotaped, providing a permanent record of the students' interactions with the calculator.

Analysis

Protocols were produced for each pair, and these were analysed using a framework adapted from that used by Schoenfeld (1985). This macroscopic analysis, represented diagrammatically using episode and time-line diagrams, allowed behaviours of interest associated with working with functions in a graphing calculator environment to be observed, contrasted and compared, and then considered more microscopically.

Results

As type and model of graphing calculator used may impact on actions available to students and can contribute to their understandings, the specific features of the calculator used and the methods available to students are described. This allows readers to compare and contrast features of other brands and models and their potential impact on student understanding in similar learning situations.

All five pairs eventually successfully produced a sketch of a complete graph of the function; however, the routes to this differed in directness and duration and the accuracy with which key features were identified. The following had the potential to facilitate the sketching of a complete graph:

1. graphing calculator features dedicated to transforming the viewing window;
2. directly altering window settings;
3. graphing calculator features enabling key features of the function to be identified;
4. linking of mathematical and graphing calculator knowledge;
5. planning; and
6. use of the table facility.

These factors may have been undertaken singly or concurrently by the students as they went about their solution.

Transforming the viewing window

The particular graphing calculator used allows the transformation of the viewing window via the ZOOM menu. The potentially most useful options for solving the task are *Zoom Box* (zooms in on a user specified area), *Zoom In(Out)* (zooms in (out) by a (default) factor of four on a user specified point), *Zoom Standard* (sets the window at $-10 \leq x \leq 10$, $-10 \leq y \leq 10$, and the scale marks to one), and *Zoom Fit* (selects y values that include the maximum and minimum y values of the function in the specified domain). Depending on the selections from ZOOM menu, the view of the function displayed may or may not change. To use these features effectively, students need an understanding of the relationship between the complete graph of a function and the viewing window displaying it.

In attempting to access the task the students used a variety of processes and their choice of these affected how successful this was initially and how quickly they found a complete view of the graph (see Brown, in press). The selection of Zoom Fit or Zoom

Standard followed by Zoom Out allowed the finding of a complete view of the graph to become potentially routine when used at the beginning of the task.

Directly altering window settings

The WINDOW menu allows direct specification of the viewing window domain and range. The scale marks can also be independently specified. The students tended to use window settings that kept the axes centred in the viewing window. A possible explanation for this may be that their prior experiences were dominated by situations where the axes were centred and therefore they found duplicating this situation simpler in terms of using the axes as a reference point for the graphical representation of the function.

Used in isolation, or in an apparently arbitrary way, altering the viewing window via WINDOW was relatively ineffectual when used by the study participants. However, when used in conjunction with other factors, particularly effective planning and linking of calculator and mathematical knowledge, this tactic potentially facilitated the determination of settings for a view of the complete graph. The first two windows shown in Figure 2 are consecutive windows viewed by Kate and Pete suggesting ineffectual alterations to the WINDOW settings with little evidence of integration with their mathematical knowledge. In contrast, the next two windows that were viewed consecutively by Abdi and Hao show evidence of mathematical knowledge and planning being used in conjunction with the alteration of WINDOW settings. It was inferred that this pair was able to link the rectangular screen output with their expected mental image of the function.

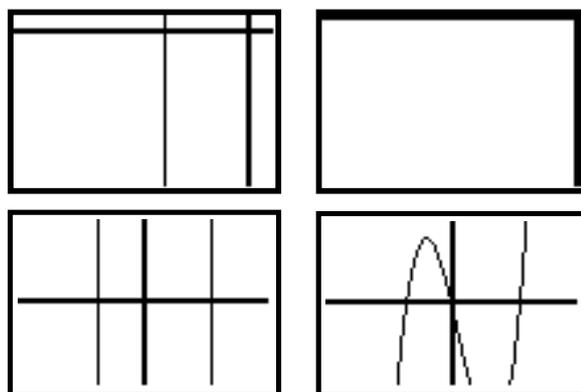


Figure 2. Effects of altering the WINDOW settings in isolation or in conjunction with other factors.

Calculator features enabling identification of key features

Once the graphical representation of a function has been displayed, additional calculator features can be used to determine key features on the graph or to assist in determining an appropriate window for a complete view. This information can be

accessed via the free cursor, TRACE, CALCULATE or TABLE, with varying degrees of accuracy and efficiency.

For the particular model of graphing calculator used, when the user is accessing the graphical representation of a function via the viewing window, the four arrow buttons on the keyboard control a *free cursor* and allow the coordinates of any point in the viewing window to be displayed. However, this cursor is not *connected* to the function. To more accurately determine the estimated coordinates of any point on the curve, TRACE can be used. The cursor is fixed on the curve in this case. Additionally, CALCULATE menu options allow more efficient identification of the coordinates of key features. Those of interest to this study include: Value, Zero, Minimum, and Maximum as shown in Figure 3(a). The results of using the first three of these are shown in Figures 3(b), (c), and (d), respectively.

MATHS: Solver provides an alternate way to determine the x intercepts of the function. The user specifies a search domain and a guess, and then the coordinates of the x intercept closest to the guess — if a zero exists in this domain — is displayed. Figure 4 shows the graphing calculator output as Ahmed and Linh accessed MATHS: Solver to correctly determine the coordinates of the central x intercept.

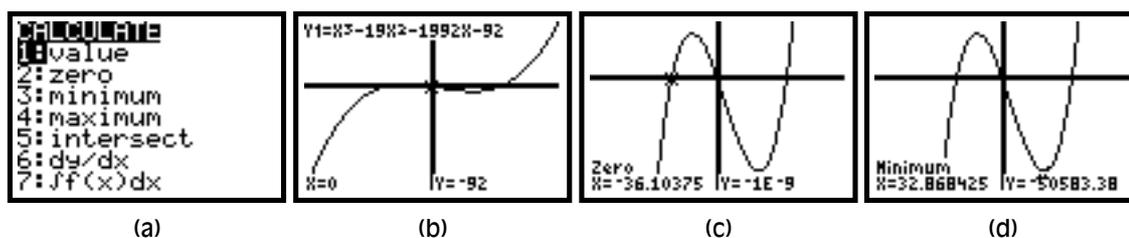


Figure 3. The CALCULATE menu and the results displayed after accessing each of the first three menu items.

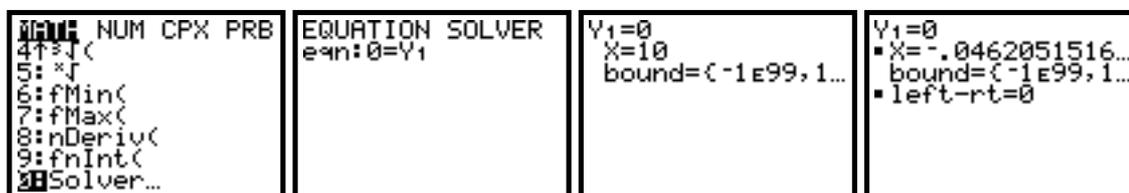


Figure 4. Using MATHS: Solver to determine the coordinates of an x intercept.

The *home screen* can also be used to determine coordinates of the function by specifying x values to display the corresponding y values. This method was used by Jing and Susan in their search for x intercepts of the function (Figure 5) by looking for a

value of x such that $y_1(x) = 0$. The TABLE also provides a method for determining the coordinates of key features. This is discussed later.

$Y_1(1)$	-2102
$Y_1(2)$	-4144
$Y_1(6)$	-12512
■	

Figure 5. Using the graphing calculator to search for zeros of the function specified by y_1 .

While students had a range of calculator features to select from in order to determine the coordinates of the key features of the function, the use of items from the CALCULATE menu best facilitated the determination of these when attempting to sketch a complete graph of the function.

Linking of graphing calculator and mathematical knowledge

The benefits of students’ linking their mathematical and graphing calculator knowledge is demonstrated by their responses when confronted with no visible section of the graphical representation in the viewing window (Figure 6).

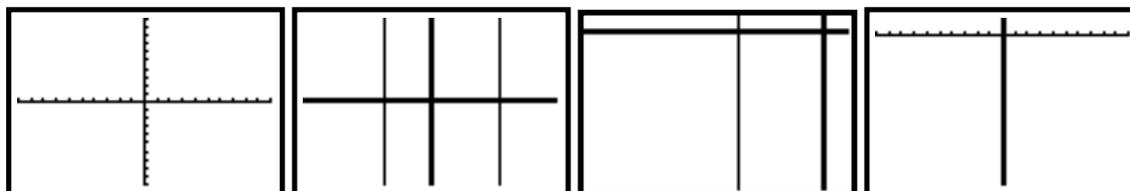


Figure 6. A view, in the standard window, showing no apparently visible portion of the graph followed by the views of three different pairs after their response action.

Actions involving adjustment of the WINDOW settings can be both physical and cognitive. These actions can reflect the students’ using either the algebraic or graphical representation of the function to inform their WINDOW adjustments and these may, in turn, be informed by the students’ understanding of the features of the calculator and their mathematical knowledge. Actions involving both cognitive and physical behaviours occurred when the viewing window was adjusted by students using their mathematical knowledge; for instance their knowledge of the y intercept or the possible shape of the function clearly informing adjustments of WINDOW values for some pairs in the study. An example of the latter occurred when Jing and Susan, in response to seeing the function as three apparently vertical lines (Figure 7(b)), altered only the WINDOW settings that affected the viewing range (Figure 7(c)), successfully identifying the position of both turning points. It is apparent that the visual

information provided by the graphing calculator (Figure 7 (b)) did not conflict with their mental image of the expected complete view of the function.

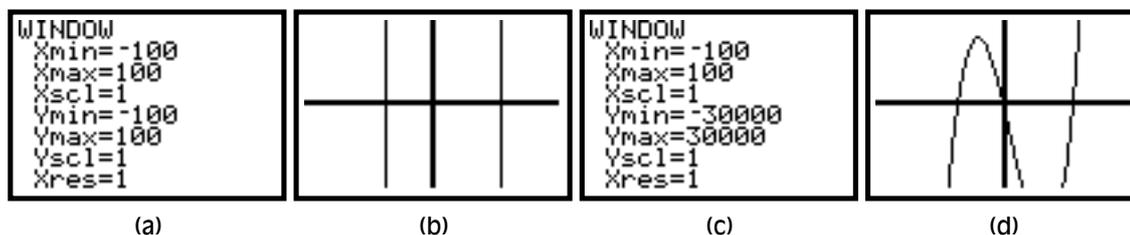


Figure 7. The vertical line effect as seen by Jing and Susan and the almost global view resulting from their informed alterations to the WINDOW settings.

Errors using CALCULATE menu items include those where an error message is displayed; for example, attempting to use value when a point is outside the viewing domain or specifying a domain that does not include a zero of the function being considered, and errors not flagged by the calculator. An example of this latter error type occurred when Abdi and Hao incorrectly selected Minimum when trying to identify the coordinates of the maximum turning point (Figure 8).

The calculator searches for the minimum (or maximum) value of the function within the specified domain which may be the endpoint of the domain, as in this case. Hence, these features need to be used carefully when determining coordinates of stationary points as the calculator is not searching for a point of zero gradient as students might expect.

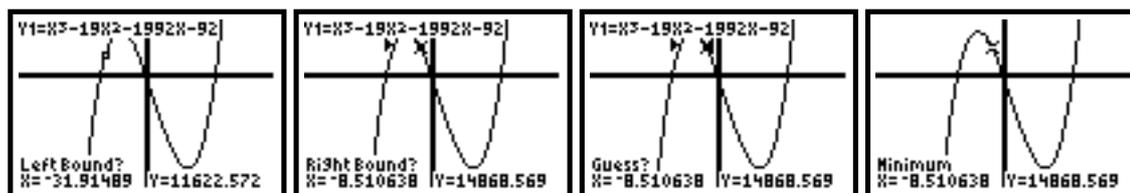


Figure 8. Incorrectly using the CALCULATE menu item Minimum to identify coordinates of the maximum turning point of the function.

Planning

In reviewing students' planning behaviours it must be borne in mind that the absence of overt planning behaviour does not exclude the possibility of covert planning. The presence of good overt planning can be inferred from students making the most judicious choices at opportune times in their solution as events unfolded, or their being able to proceed immediately with the task without an obvious need to stop and plan as they had the mathematical skills in their repertoire to efficiently solve the task by

applying routinely used procedures. When difficulties arise, good planning is indicated by students taking a step back in their solution path to make a considered alternate choice, whereas bad planning is demonstrated when a current pathway is abandoned without evaluation and students select alternate, or return to their initial, choices without reflection on what they have just done.

Good planning was exhibited by the Year 12 pair, Linh and Ahmed. They found a perfect global view, from their perspective, before proceeding to identify any key features of the graph. They were the only pair whose behaviour fell into two distinct categories: the finding of a good global view followed by identification of key features. This pair demonstrated integration of graphing calculator skills seamlessly into their mathematical routines.

In contrast, the other Year 12 pair demonstrated poor planning early in their solution process. Although they found a global view of the graph after one minute, as they began identifying the key features they became dissatisfied with this view, as some of these were barely distinguishable from the axis. In attempting to improve the view, they demonstrated little planning to facilitate progress. They knew about Zoom Fit: its use in conjunction with sensible viewing domain adjustments would have more efficiently produced an improved global view rather than what they did. They took several backward steps in their solution process, proceeding to only alter the WINDOW settings as they sought an improved view of the function.

The Year 11 pairs varied considerably in the amount and type of planning undertaken. One pair spent a considerable time when they undertook minimal discussion but made innumerable WINDOW adjustments often with little effect on the view of the graph. Their actions were neither assessed nor varied. None of the other dedicated functions of the graphing calculator were used. In contrast, a second pair spent a significant amount of time and discussion considering each change they made to the WINDOW settings, pursuing one particular solution strategy before they changed approaches and repeated this process for another solution strategy. The final pair, after finding themselves very close to a global view of the function early on and, as a consequence, the solution process having the potential to become routine, exhibited poor planning as they lost their 'almost' global view (Figure 7(d)) and almost lost sight of the function altogether.

Use of the TABLE facility

Once the algebraic representation of a function has been entered, coordinate pairs can be viewed via the TABLE. By pressing TABLE SETUP, ΔTbl (the difference between consecutive x values) or $TblStart$ (the initial x ordinate displayed) can be adjusted as shown in Figure 9. Judicious selections allow the user to find coordinates of any points on the curve such as the value of a y intercept, as in the first table in Figure 9, or where a turning point can be inferred to exist, as in the second table. Further adjustment to the TABLE SETUP would allow the coordinates of this turning point to be more accurately identified. By sensible application of mathematical knowledge the coordinates of any key feature of the function can be determined.

TABLE SETUP		
TblStart=-.5		
ΔTbl=.5		
Indent: Auto Ask		
Depend: Auto Ask		

X	Y1	
-1	-1.25	
0	-2	
.5	-1.25	
1	1	
1.5	4.75	
2	10	
2.5	16.75	
X=-.5		

X	Y1	
-1	1	
0	0	
1	1	
2	4	
3	9	
4	16	
5	25	
X=-1		

Figure 9. Altering the initial values and the difference between consecutive terms using TABLE SETUP and TABLE outputs where key points can be identified or inferred.

A judicious choice of value selection for ΔTbl and $TblStart$ can inform appropriate WINDOW settings to facilitate the viewing of a complete graph. While this would have been a sensible action for students in this study, as they experienced difficulties identifying the WINDOW that provided a global view of the function, only one pair considered this course of action. Kate suggested the table be used, but this was initially rejected by Pete as cheating. After she repeated the suggestion, his argument that the graphical rather than numerical representation was better given the situation, resulted in their not using the TABLE.

Discussion

The key to effective and efficient use of the WINDOW menu is understanding how selection of each option alters the window settings and the resultant effect on the view of the graph. Determining which menu item to select to adjust the view of the graph in any required way is neither simple nor intuitive. To become expert in the correct selection, most students require significant experience in observing the effects of their use. The students must position the current output of the graphing calculator screen with their mental image of the cubic function; as for many of the rectangular outputs displayed by the calculator, there are myriad positions this could correspond to in one's mental image of a cubic, or other, function.

In addition, students need to be observant in their use of the CALCULATE menu. The incorrect use of any item will result in an incorrect answer, and unless students take notice of the graphing calculator display they may draw incorrect conclusions about coordinates of the function.

Conclusion

While it was apparent in this study that both calculator and mathematical knowledge are essential for students to solve a problem task in a graphing calculator environment, it is the judicious choice of the calculator features linked with appropriate mathematical knowledge and good planning that allow effective, efficient, and accurate task access and solution.

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What is 'Working Mathematically'?*

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Curriculum documents stress the importance of 'Working Mathematically'. At the same time, there is an increasing demand for accountability, measured by state-wide multiple-choice tests. This creates tension between curriculum expectations and system demands.

Addressing the improvement of basic skills through Working Mathematically approaches emphasises deep rather than shallow learning. The activities discussed in this paper will address the development of a range of mathematics skills through open-ended problem solving activities suitable for students from grades 6 to 10. The activities will be drawn from different areas of the mathematics curriculum.

Introduction

'Working Mathematically' is a term used in Australia to encapsulate what is believed to comprise the processes of doing mathematics. The inclusion of the Working Mathematically strand in all Australian state syllabi is in concert with a trend in many countries that adopts the view that learning mathematics involves more than the acquisition of a body of facts and procedures transmitted from teacher to pupil. The *National Statement on Mathematics for Australian Schools* (Australian Education Council (AEC), 1991) identified three domains of working mathematically: *Attitudes and Appreciations*, *Mathematical Inquiry* and *Choosing and Using Mathematics*. Each of these domains described processes that contribute to the learning of mathematics but are not necessarily bound to mathematical content. These processes include observation, generalisation, pattern representation, the use of mathematical notation, terminology, conventions and models, the justification of insights, and the cultural contexts of mathematics.

* This paper has been subject to peer review.

All states and territories now include some form of Working Mathematically outcomes in their mathematics curriculum, although with some differences in terminology. For example, Tasmania has adopted a *Working Numerately* approach with two sub-strands of 'Posing and Answering Questions' and 'Thinking About Reasonableness' (Department of Education, Community and Cultural Development, 1997). In contrast, the Victorian *Curriculum Standards Framework* (CSFII) has a global description of Working Mathematically with staged outcomes in 'Reasoning and Strategies', divided into sub-strands of 'Mathematical Reasoning' and 'Strategies for Investigation' (Victorian Curriculum Assessment Authority, 2002), and New South Wales has adopted six sub-strands of Working Mathematically, including 'Questioning', 'Problem solving' and 'Communicating' (Board of Studies NSW, 2002). There is a considerable degree of similarity. Most state documents, somewhere, state that processes involving solving problems and developing suitable strategies, communicating mathematics clearly and unambiguously, making conjectures, and drawing logical and justifiable conclusions are important aspects of children's mathematical learning. Emphases differ, as does a philosophical approach as to why working mathematically is important. In some documents it is clear that working mathematically is seen as an aid to the development of a 'disposition' towards using mathematics. In others it is seen as a further set of skills to be learned.

Why, then, do teachers not use Working Mathematically approaches? Teachers do appear to find teaching children to work mathematically problematic. There are two possible reasons for this. The first is that clear descriptions of what working mathematically is have not been articulated. Working mathematically is understood through descriptions of what children are believed to do. Even so, confusions remain. To give one example, in *Mathematics – A Curriculum Profile for Australian Schools* (AEC & Curriculum Corporation, 1994) 'Working in Context', a sub-strand of Working Mathematically, is written to mean the development of a socio-historical perspective of mathematics. In other state documents, this is interpreted as using 'real world' situations to develop or practise mathematical concepts.

The second problem arises when teachers need to assess children's progress or achievement of certain Working Mathematically outcomes. There is little research as to when children may demonstrate particular mathematical behaviours, or the extent to which these behaviours are related to underlying conceptual development. To illustrate the point, it is not clear, at least in NSW, whether children's unsophisticated explanations, and informal or idiosyncratic symbols or diagrams, are acceptable indicators of mathematical progress, or whether only the use of formal mathematical conventions can be accepted as valid indicators of mathematical development. This is compounded when state-wide tests use only formal approaches in their construction.

Accountability processes often do not stress Working Mathematically outcomes. For pragmatic reasons, the focus is on aspects of mathematics that can be easily tested and machine marked. Of necessity, these become relatively low-level content. In Australia, *National Benchmarks for grades 3, 5 and 7* are aimed at the bottom 20 percent or so of students – those who might experience difficulty in school if they have not achieved the benchmark standard (Curriculum Corporation, 2000). Consider the benchmark standard for Number at Year 5:

Students are developing their knowledge of numbers to include larger whole numbers and simple decimal fractions in familiar situations. They are also able to deal with some simple common fractions. They make choices about whether to add, subtract, multiply or divide, and can decide which is the best method to calculate — whether to calculate mentally or use a written method or a calculator — and can use any of the methods accurately. (Curriculum Corporation, 2000, p. 4)

Working Mathematically ideas are suggested by the sense of students choosing the mathematics needed, and the appropriate tools to use for a particular situation, but there is little of the rich and exciting approach to numeracy that Working Mathematically orientations can supply.

Does this mean that teachers cannot use the kinds of approaches to teaching mathematics that are essential to foster Working Mathematically? If students are to do well on statewide testing programs, surely they need to know what to expect and to have opportunities to practice their skills? No teacher will ever willingly jeopardise students' chances on any external assessment. This is the reason for the 'ripple effect' of high stakes assessment (Stephens, Clarke, & Pavlou, 1994). So how can teachers combine the best of both worlds and provide a rich Working Mathematically environment as well as ensuring that students' skills are adequate for testing? In other words, how can basic skills practice and Working Mathematically be combined to produce a productive working environment?

This paper describes and discusses examples of classroom activities that can be used not only to develop children's understanding of mathematical ideas, but also to develop their mastery of mathematical skills, as well as cultivating their consciousness about the processes involved in doing mathematics. This does not imply that teachers need develop complex and difficult problems. Rather, we want to show that a shift in the focus of questions can open up the learning experience.

Questions asked in the classroom are, broadly speaking, of two types. There are the questions that set the problems for students to investigate and solve. Often these questions are 'closed'. They focus students on obtaining an answer: for example, 'If a soft drink costs \$2.30, a packet of crisps \$0.90 and a bar of chocolate \$1.20, how much change will Susie receive out of \$5.00?'. The problem can be made more complex by supplying more data or creating more questions using the data. Once students find an answer, they move on to another 'problem'. The basic skills of arithmetic operations are being used, but they are being used in a purely procedural manner. This is a problem that students can identify by type and simply switch on the required procedure without necessarily understanding why they are carrying out the routine. To encourage a richer approach, Sullivan and Clarke (1987) suggest opening up such questions. This broadens the focus to encompass some analysis and discussion of processes and the justification for several possible solutions. To use the shopping example above, rather than the closed question illustrated, a teacher could provide students with a price list of various items and ask questions such as, 'What could you buy for \$5.00? What is the greatest number of items you could buy? If you wanted to save one-fifth of your money, what could you buy now?'.

The other classroom questions are those that teachers ask of children as they encourage students to reflect on what they have done in the course of solving a problem or

conducting an investigation. Hiebert and associates (1999) use the term ‘problematising mathematics’. Questions such as, ‘Why did you do that?’, ‘How do you know?’, ‘What other ways did people in the class do it?’ can be applied to mathematical situations such as having children mentally add two numbers and then describe how they carried out the computation, either verbally or in writing.

This paper aims to describe two practical classroom situations that are rich in both Working Mathematically opportunities and basic skills practice. We take the view that these two approaches are not incompatible, and that developing positive attitudes to mathematics, thinking skills related to mathematics and confidence when doing mathematics, while drawing on a range of mathematical tools, will provide a more than adequate basis for approaching testing programs, as well as fostering deep mathematical learning. None of the activities are new, and all are readily available. Rather, our intention is to exemplify and expand the Working Mathematically and basic skills links through consideration of rich teaching and learning approaches.

Scenario 1: Number operations and patterns

Eric the Sheep (Mathematical Association of Victoria, n.d.)

Eric the sheep is lining up to be shorn. There are fifty sheep in front of Eric. He can’t be bothered waiting in the queue, so he decides to sneak towards the front. Each time one sheep from the front of the line is taken to be shorn, Eric moves forwards past two sheep.

How many sheep will be shorn before Eric gets to the front?

Algebraic reasoning is a mathematical abstraction that many students struggle to relate to their experience. While *Eric the Sheep* lends itself to the exploitation of many problem-solving strategies, it also provides learning experiences that can be used to develop basic algebraic concepts.

Problem solving strategies include role playing, breaking the problem down into simpler situations, using concrete objects and using computer simulations. The whole class could model the problem through role playing. This can become as elaborate or as simple as the class or teacher’s imagination will allow. Students will be faced with the necessity to find a model using fewer sheep than the problem requires. In deciding what to do and how the results from the simpler model can answer the actual question, students will discuss their ideas and strategies, test numbers, and verify their answers. The teacher may provide guidance, if and when necessary, but it is illuminating for the teacher to simply watch and listen as the students resolve the various problems.

Other problem-solving strategies could follow. For example, students could model the problem using counters and a toy sheep; older students might construct spreadsheet programs that enable them to extend the question to much larger figures as well as deconstruct the problem and create appropriate algorithms.

Students may also conjecture about patterns that involve Eric moving two or three sheep at a time or how patterns develop if the number of shearers changes. All of these possibilities and variations need to be recorded, and then inferences drawn, tested and

verified, helping to develop and refine mathematical skills in communicating ideas and representing mathematical processes.

What are the basic procedural skills practised? In teasing out the solution to this problem students will use the four arithmetic operations, but also make connections between each operation and its inverse, between repeated addition and multiplication and discover number patterns that may be illustrated on hundreds charts and used to consolidate understanding and recall of number facts that assist in mental computation.

An appreciation of patterns and their representation by rules is a skill fundamental to algebra. Students find it relatively easy to use a given rule to create a pattern of numbers, but they need not have an understanding of the role the variables play, nor of the underlying meaning of the rule itself. They may simply follow the recipe described by the rule. Deep understanding of algebraic processes requires students to be able also to establish algebraic rules from a data set. It requires fairly sophisticated mathematical thinking to move beyond a recursive, additive or subtractive description of a pattern, such as ‘it goes up by threes’, to a description that relates all the component variables. *Eric the sheep* provides a context for students to explore a large number of patterns and so develop concepts and strategies for playing the algebraic game with understanding.

Students who have worked on problems such as *Eric the Sheep* have experience and a sound conceptual basis for dealing with questions such as those in Figures 1 and 2, taken from the Year 10 NSW School Certificate (Board of Studies, 2000).

A. Consider the pattern:

$$3^2 - 2^2 = 3 + 2 = 5$$

$$4^2 - 3^2 = 4 + 3 = 7$$

$$5^2 - 4^2 = 5 + 4 = 9$$

Use the pattern to complete

$$29^2 - 28^2 = \boxed{} + \boxed{}$$

Figure 1. Typical Year 10 algebra problem (Board of Studies, 2000, Q21, Section 1, p. 6).

x	-2	-1	0	1	2
y	-2	0	2	4	6

What is the correct rule for this table?

- (A) $y = 2x + 2$ (B) $y = x + 4$ (C) $y = x + 2$ (D) $y = 2x - 2$

Figure 2. Year 10 algebra rule problem (Board of Studies, 2000, Q53, Section 2, Part A).

The use of a mathematically rich task, *Eric the Sheep*, can lead to practice of basic skills such as multiplication tables, pattern recognition and early algebra ideas. Extension activities could include students creating their own scenarios for other class members to investigate, or writing stories given algebraic rules or patterns. Such activities develop skills in analytical reading of word problems, in algebraic modelling and in relating algebraic solutions to the context from which the problem arose.

Scenario 2: Area and perimeter

Pentominoes (Mottershead, 1977)

How many different shapes can be made from five unit squares? The squares must be joined exactly along common edges.

How many different rectangles can be made using all the pieces? What is the area of each rectangle? What is the perimeter of each rectangle?

Students enjoy the hands on nature of explorations with pentominoes. Square ‘sticky-notes’ provide concrete materials that can be moved around if more expensive modelling materials are not available. Once students think they have found all possible arrangements they can record these on squared paper.

What opportunities does this provide for Working Mathematically? Questions such as, ‘How do you know you’ve got them all?’ lead naturally to ideas of verification and proof. Asking students to develop a notation to describe all the combinations is a far from trivial task. This has challenged mathematics graduates who are pre-service teachers, and allows for the development of mathematical communication skills. Describing the transformations needed to verify whether one arrangement is the same as or different from another helps to develop the mathematical ideas of reflection and rotation, which are important bases for sophisticated geometrical understanding.

There are many extensions of the pentominoes activity but the one we have chosen to concentrate on is one that explores the ideas of perimeter and area. As an initial problem, students may be asked to find the perimeter and area of each one of the twelve pentominoes. It may not be immediately obvious to younger students that they all have the same area, especially if they see area as ‘length \times breadth’. One pentomino has a perimeter of ten units, while the other eleven all have perimeters of twelve units. Further discussion could centre on the shape of the one that is different — it is the shortest and fattest shape (the closest approximation to a circle).

Extending this investigation to the range of rectangles that can be made using all twelve shapes provides a rich source of questions that develop conceptual understanding of the difference between linear and square measurement, and to the development of the algorithm for finding the area of a rectangle. Consider the following questions:

If we make rectangles using all 12 shapes, what must the area of each rectangle be?

What are the possibilities for the side lengths of rectangles made with all 12 shapes?

Verify your predictions by using your shapes to make each different rectangle.

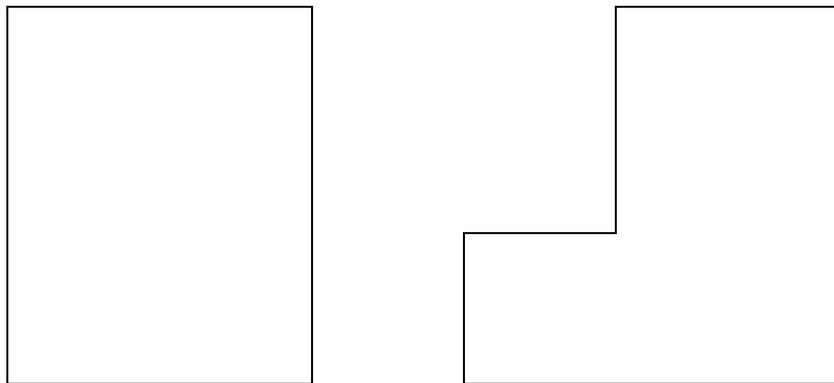
These kinds of questions lead naturally to the development of the algorithm, and subsequent generalisation to all rectangles.

More experienced students can be challenged to prove that the smallest rectangle possible is a 3×5 , or to find all the possible rectangles that can be made using a given number of shapes, or to find what squares can be made. This learning experience is much richer than, 'Find the areas of the following rectangles,' followed by several examples requiring multiplication of length and breadth.

However, this is time consuming 'play' and your principal wants good results on the test. How is this going to help? A typical test question at year 7 is shown in Figure 3, taken from work samples included in the *National Numeracy Benchmarks* (Curriculum Corporation, 2000).

You need to put carpet on the floor of two rooms in a house.

Here are the room plans.



Put a tick on the room which will need more carpet.

Explain how you know which room will need more carpet.

Figure 3. Year 7 test question (Curriculum Corporation, 2000, p. 64).

Arguably students who have experienced making pentominoes will have a grasp of the idea of putting squares together to cover surfaces, and those who have been asked to describe what they are doing will be better prepared for the second part of the question where students are asked to explain their reasoning.

Providing Working Mathematically opportunities in this instance leads naturally to a development of conceptual understanding that provides the building blocks for further development into ideas of volume and total surface area. At the same time it does not detract from the necessary procedural knowledge also needed for success.

Conclusion

We have discussed in detail two rich activities that enable children to work mathematically and, at the same time, to master essential mathematical skills. There are many tasks in resources such as the *Mathematics Curriculum and Teaching Project (MCTP)* (Lovitt & Clarke, 1988) which are readily available and provide opportunities for students to develop both mathematical content and process understanding. Some examples include *Four Cube Houses*, *Crazy Animals*, and *Today's Number Is*. Activities such as *Sphinx* (Williams, 2000) and the MCTP *Map of Australia* could be used to consolidate understandings developed through the use of *Pentominoes*. The MCTP *Algebra Walk* and *Mind Reader* activities enable further development of concepts introduced by *Eric the Sheep*. These ideas have been around for many years now, and the Internet is now another source of rich problems. Indeed, many of the activities quoted here are available from the *Maths 300* website (Curriculum Corporation, 2002).

As teachers we should not be diverted by the low-level demands of procedural tests. Instead, let us use the good practice we know about to provide students with opportunities to practice basic skills while working like mathematicians.

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Problem formation on the Australian Mathematics Competition*

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This paper will describe the role played by a teacher on the AMC problems committee. The teachers on the committee have the roles of classifying groups of questions into divisions ensuring syllabus fairness, looking at statistics from previous years and making judgements on curriculum issues. These curriculum issues include gender fairness, language, relevance and values of each question. The paper will point to the educational uses of the AMC and will provide some anecdotes about some of the papers' more interesting questions.

Background to the AMC

The Australian Mathematics Competition for the Westpac Awards (AMC), modelled on the USA and Canadian computer-processed mathematics competitions, started in 1976. By 1992, with over 500 000 entries, it has become one of the world's largest mathematics competitions.

The Australian AMC entries are equivalent to over 30% of all secondary school students in Australia (or 1 in every 48 Australians, i.e. nearly 3% of the total 16 000 000 population of Australia)

Aims of the AMC

The aims of the AMC are:

- to encourage secondary school students to strive to the best of their ability in mathematics, that is, a personal pursuit of excellence at an early age in their studies;
- to encourage mastery of the basic numerate skills; and
- to create, over the years, a pool of interesting, rewarding and challenging problems to help extend, supplement and enrich school work.

* This paper has been subject to peer review.

The competition papers are not designed specifically for students who have high innate ability in mathematics, even though such students naturally do well. It has been found that good, conscientious students of average ability find many questions within their experience and competence while other questions interest and challenge them.

Educational use

The AMC can be a useful stimulus not only for challenging students of different levels of ability but also can be useful for the teaching and learning of mathematics. In other words, the wide range of statistical information from the AMC can be a useful source of discussion in department meetings within a staff-room. The information is in the form of school, regional and national response rates to each question by year group. Teachers can discuss topic groups, year group comparisons, distractors and any given student's answers. The solutions and statistics booklets contain a wealth of information for teacher professional development.

Mathematics staff may wish to 'match' questions against the state or school programs; for example, 'This topic was covered by Year 9 just before the AMC, while Year 10 had not done it for 6 months'. Further, some staff are interested in ranking students on their school performance and comparing the outcome to AMC rankings (correlations).

Analysis of questions

The response rates for questions are of interest in their own right and are used both in the setting of future competition papers and for research in the learning of mathematics. (For example, in the last ten years, the AMC has a data bank consisting of responses and other information for nearly five and a half million entries.)

Good discriminators

A question is a good discriminator if the more able students successfully attempt the question. In order to determine questions which are good discriminators, a statistic called the *discrimination biserial* is used.

This is a number which, for each question, may take a value between -1 and +1. If negative, the statistic means that more students in the lowest 27% ranking in the competition obtained more correct responses to the question than the students in the top 27% ranking. A question is usually regarded to be a good discriminator if it has a discrimination biserial of at least 0.4.

It has been found that the AMC's best discriminating questions either test very basic syllabus skills or are descriptive questions which are not too difficult but require modelling skills and do not have a diagram given as part of the question. The questions that come out as high biserial values are often surprising. Consider the following examples:

Question 10: (school years 7 and 8, 1981)

If $y = x^2 + 2x = 3$, the value of y when $x = 3$ is

- (a) 5 (b) 15 (c) 21 (d) 18 (e) 0

	Percentage Response Rate					
School Year	A	B	C	D	E	Omit
7	8	12	6	57	3	14
8	4	7	3	78	2	6

The discrimination biserials here were 0.69 and 0.74 respectively for school years 7 and 8 and so is an extremely good discriminating question. This question specifically tests the basic skill technique of substitution in a formula.

Question 7: (school years 7 and 8, 1989)

If the greatest and least of the numbers 0.31, 0.303, 0.5, 0.675 and 0.68 are added, the sum is

(a) 0.99 (b) 0.985 (c) 0.983 (d) 0.978 (e) 1.175

	Percentage Response Rate					
School Year	A	B	C	D	E	Omit
7	5	6	36	10	30	13
8	6	5	46	9	25	9

The discrimination biserials here were 0.53 and 0.56 respectively. Students had to be able to first determine what numbers were the greatest and least before they were able to determine the sum. The high response rate for alternative (e) is of interest. It is obtained by adding 0.675 and 0.5, that is, the decimal with the largest set of digits and the shortest set of digits. The response rate for (e) surprised many of the committee.

Process of problem formation

Each member submits

- solutions
- statistics: gender, discrimination, Year 9/10 etc.
- wording
- chair
- distractors
- classification bands
- state syllabus
- state moderators
- balance of topics
- length
- fairness of distractors
- diagram vs no diagram
- common questions
- language
- further enrichment.

Problem solving strategies

This is not an exhaustive list.

1. Read it carefully; understand what the problem asks. Hurried or careless reading is often the source of wrong answers.
2. Re-state the problem.
3. Guess and check or trial and error.
4. Make a table, chart, set of lines, group of boxes etc.
5. Do not rush.
6. Look for patterns.
7. Do not rub it out, cross-out lightly.
8. Draw a diagram, model or picture, tree-diagram.
9. Eliminate possibilities.
10. Work backwards.
11. Keep writing.
12. Organise data; have a good system of recording.
13. Use logic, e.g. 'What if...?'.
14. Act out a situation, e.g. 'stepping' with your feet or hands.
15. Solve a similar, simpler problem.
16. Cut or fold up pieces of paper.
17. Estimate.
18. Remove extraneous information. Some things may have nothing to do with the answer or solution.
19. Is it an extension of a problem I have seen before?
20. Check language: look for keywords, e.g. successive, consecutive, less, greater, original, initial, if, unless, symmetrical, equal, double, etc.

The future

Despite the possible emergence of other competitions, the AMC will continue to provide credible, authentic information and statistics as well as enrichment material for schools. The questions will continue to go through a thorough moderation process.

Against the tide — Calculator-free mathematics

Michael Cody

Camberwell Grammar School, Vic.

In Victoria, students are learning to be proficient in mathematics through their calculators. The number and complexity of some of the programs that they can utilise in the examinations via their programmable graphing calculators is quite astounding. Consequently, calculator skill development is becoming more and more prominent in junior classes.

My school has resisted having calculators on the book list until Year 9. Our students have not been disadvantaged. In fact, our success in state and national mathematics competitions would suggest just the opposite — but then, the competitions tend to be calculator free.

In this session, we share some tricks of arithmetic that can be used in upper primary and all levels of secondary mathematics classes.

Is it not frustrating when you are teaching Year 12 and see students reach for their calculators to work out the answer to 6×10 or $27 \div 3$? This is a symptom of calculator reliance that has become endemic from late primary and early secondary school mathematics. I am a firm believer that a calculator should only be used as a regular tool for arithmetic if, by Year 8, the student has been proven incapable of learning their tables up to '10' and has no concept of place value.

Do not get me wrong: I am not anti-technology. Far from it! The use of a calculator or computer to investigate and extend a concept in mathematics can be quite wonderful. However, once the concept is understood, we can move on and put those tools away for when they are again an 'essential' item.

For arithmetic, I do not see that calculators are 'essential' items. Nearly every school I have taught in since 1977 has been pro-calculator; students have been expected to purchase one in Year 7 and they can get right on with using them, getting the 'correct answers' — provided the correct buttons are pushed and extra brackets are employed when necessary. Confidence with numbers is quite empowering and it is interesting to see the reactions of classmates to a person who can quickly produce a correct response to a multiplication that generates a four or five figure result.

At Camberwell Boys Grammar School, calculators are not on the book lists until Year 9. Boys come to the school from many different primary schools. Our numbers swell from

40 students in Year 6 to 175 in Year 7. The mathematical backgrounds of students are enormously different and we block and stream our classes at the end of the first semester.

The initial emphasis is purely on number skill: from integers, through decimals, fractions and percentages – without calculators. We go through the Year 7 and 8 syllabuses ‘almost’ entirely without calculators. Pythagoras is introduced in Year 8 and, because we cover surds in Year 9, a calculator becomes convenient to determine square roots. During this topic, we ask the boys to bring in ‘any’ calculator that has a $\sqrt{\quad}$ button. Most boys can find one at home. We get some wonderful calculators come in, from 10 cm \times 10 cm ones with big buttons, some that look like a banana, as well as TI-92s. We have a page or two of square root tables for those who do not bring a calculator, as I had when I was a student. (Therein initiates another useful skill that is being lost, the ability to read tables!)

It was actually discussing non-calculator determination of ‘square roots’ with a colleague that inspired me to do this presentation. How many current mathematics teachers know how to determine a square root without a calculator? Not too many, I would venture to guess.

The colleague was a current, but very mature, English teacher and he reminded me of a routine he used to use (which I had also been shown when I was in form 2). This is a routine that I have since re-discovered in a mathematics text published during the Second World War, and also in a mathematics text published in 1729 (having belonged to Robert Browning’s grandfather and loaned to me by the aforementioned English teacher). By revisiting this algorithm, it made me think about just how much we are denying the youth of today some of the wonderful mathematics that empowered us and our own teachers, and their teachers.

There is no question that we must ensure our students are efficient with calculators and other forms of technology by the time they get to their VCE, but I believe that we should not have to sacrifice basic number principles to achieve this end.

In the last five years, Camberwell Grammar has produced over a dozen students with the highest possible study scores in their final year of mathematics. We consistently achieve in the order of twenty prizes in the AMC. Boys’ projects have gone on to national finals for the Mathematics Talent Quest and we have also had champion teams for State Mathematics Problem Solving/Games Days at Years 7, 8, 9 10 and 12 levels. I attribute this success to ensuring that our boys are stretched with their numeracy and problem solving skills and do not become just efficient button pushers.

One way of reducing calculator reliance is to increase the students’ confidence with basic arithmetic. I will demonstrate just a few fascinating and time saving techniques that are not your standard procedure.

Finger reckoning is the art of arithmetic calculation on your fingers. Most people these days know the 9-times table (up to 10) by finger reckoning. Well what about 19×17 ? Or 31×34 ? Or 26^2 (great for Pythagoras)?

There is quite a delightful algorithm for the multiplication of numbers within the same tens group but unit groups of 1–5 and 6–0, (it gets a bit more complicated if we go across groups, e.g. 36×52).

For two values with units in the 1–5 group it goes like this:

For 31×34 , both are in the 31–35 group.

The lowest tens value of this set of numbers is 30 and the highest is also 30.

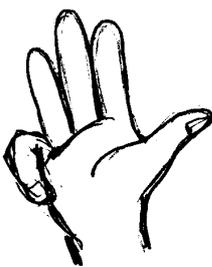
Multiply the lowest tens value by the highest: $30 \times 30 = 900$.

Left hand fingers count:



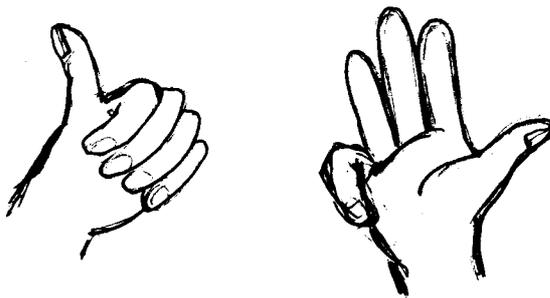
31 — one finger up!

Right hand fingers count:



31, 32, 33, 34 — four fingers up!

Five fingers up altogether, multiply by highest tens value (= 30): $5 \times 30 = 150$.



Hands for 31×34 .

Multiply fingers UP from each hand: one (left hand) \times four (right hand) = 4.

Answer is: $900 + 150 + 4 = 1054$.

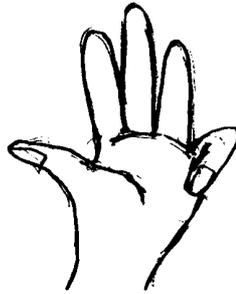
For two values with units in the 6–0 group it goes like this:

19×17 , both are in the 16–20 group.

The lowest tens value of this set of numbers is 10 and the highest is 20.

Multiply the lowest tens value by the highest: $10 \times 20 = 200$.

Left hand fingers count:



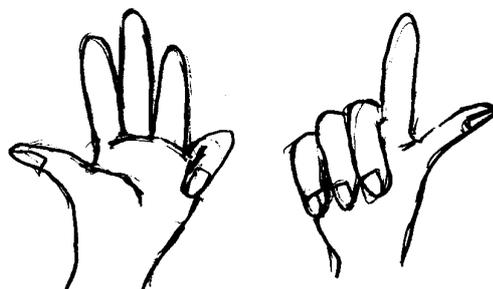
16, 17, 18, 19 — four fingers up!

Right hand fingers count:



16, 17 — two fingers up!

Six fingers up altogether, multiply by highest tens value: $6 \times 20 = 120$.



Hands for 19×17 .

Multiply fingers DOWN from each hand: one (left hand) \times three (right hand) = 3.

Answer is: $200 + 120 + 3 = 323$.

You might think it would be quicker to use pen and paper, but you do get used to the calculating — and it is a bit of fun (colleagues will watch you with amazement!).

To handle the sums that go ‘cross’ groups like 35×42 , you could treat it like $35 \times 32 + 35 \times 10$. Similarly 27×22 is $25 \times 22 + 2 \times 22$.

These obviously get messier and require a good understanding of the commutative properties of multiplication.

Squaring numbers is easy with finger reckoning: e.g. 43^2 is $1600 + 240 + 9 = 1849$.

However, another colleague prefers to use the binomial expansion of $(a \pm b)^2$ to calculate the answers to squares:

$$43^2 = 40^2 + 2 \times 40 \times 3 + 3^2 = 1600 + 240 + 9 = 1849.$$

Let us try the upper group: e.g. 27^2 .

By using binomial expansion,

$$27^2 = 30^2 - 2 \times 30 \times 3 + 3^2 = 900 - 180 + 9 = 729, \text{ or}$$

$$27^2 = 25^2 + 2 \times 25 \times 2 + 2^2 = 625 + 100 + 9 = 729.$$

By finger reckoning, $27^2 = 20 \times 30 + 4 \times 30 + 3 \times 3 = 600 + 120 + 9 = 729$.

Vedic mathematics¹ has elegant shortcuts for two and three digit multiplication as well.

Of particular interest is the multiplication of numbers close to 100, e.g. 88×93 .

88 is 12 off 100 and 93 is 7 off 100. Subtract 7 from 88 to get 81 (leading two digits of the answer) and multiply 12 by 7 to get 84 (last two digits of the answer).

$$88 \times 93 = 8184.$$

Similarly 109×107 is 11 663, the 116 = $109 + 7$ or $107 + 9$, and the 63 is 7×9 .

Another ‘instant answer’ technique is provided for the square of any two or three digit number ending in 5. The answer always ends in 25, and the leading digits of the answer are the product of the leading digits of the original with ‘one more’:

$$\text{e.g. } 35^2 = (3 \times 4)25 = 1225, 115^2 = (11 \times 12)25 = 13225.$$

Now, for those square roots:

Let us just go ‘correct to one decimal place’ — but we can keep going if we want to!

Consider $\sqrt{1074}$.

Pair off values either side of the decimal point: 10 74 . 00 00.

What is the closest square root under the left most pair? Answer: 3. Subtract the square of this and bring down the next two digits to get 174.

¹ Vedic mathematics is the name given to the ancient system of mathematics which was rediscovered from the Vedas between 1911 and 1918 by Sri Bharati Krsna Tirthaji (1884–1960) — it is not some sci-fi trekker maths! More interesting details can be found at <http://vedicmaths.org>.

Double the first value ($2 \times 3=6$) and find the highest 60s (in this case) that goes into 174. Only 2 will go, so the second digit of the answer is 2. $174 \div 62 = 2$ and 50 remainder.

Bring down the next two digits, double the 2 value, put it into the tens column, with the original value going into the hundreds. Now we are looking for 640s into 5000, which goes 7 times. $5000 \div 647 = 7$ and 471 remainder. (At this stage we have $\sqrt{1074} = 32.7$, but looking like 32.8, as 471 is a majority of 647.)

Bring down the next two digits to make 47100. Move 64 across one more place, double 7 and add into the tens column, making 654_.

Now, how many 6540s go into 47100. Answer 7 (this is the 100ths place in the answer). i.e. $\sqrt{1074} \approx 32.77$, therefore $\sqrt{1074} = 32.8$ correct to one decimal place.

The working out for this is pretty routine for someone who has a good understanding of order of magnitude and is skilful with multiplication. It is not the sort of thing that is easily done in one's head:

$$\begin{array}{r}
 32.7 \\
 3 \overline{)1074.0000} \\
 \underline{-9} \\
 62 \overline{)174} \\
 \underline{-124} \\
 647 \overline{)5000} \\
 \underline{-4529} \\
 6547 \overline{)47100} \\
 \underline{-45829} \\
 6554 \overline{)127100}
 \end{array}$$

3 is the highest root into 10.

Take away the square of 3.

Bring down next 2 digits. Double 3, place in tens place and see how many 60s go into 174. Answer 2. Does 62 work? Yes. Use 62.

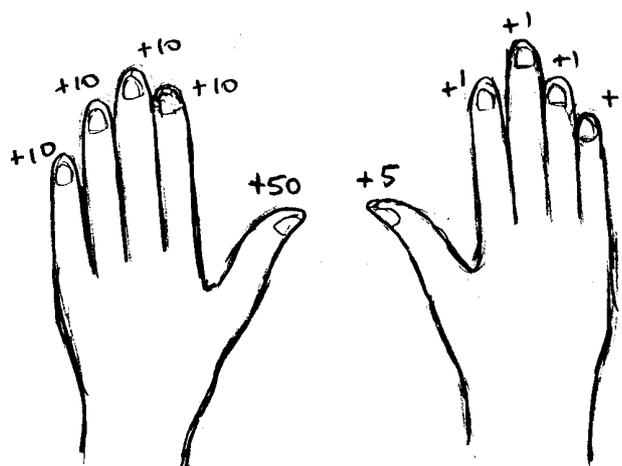
Subtract 2×62 and bring down next 2 digits. Double the previous 2 and push along into tens place. How many 640s go into 5000? Answer 7. Does 647 work? Yes. Use 647

Subtract 7×647 and bring down next 2 digits. Double the previous 7 and push into tens place. How many 6540s go into 47100? etc.

The principle behind the result is based on a binomial expansion of a square, but in reverse. Cube roots are similarly done using binomial expansion of a cube, in reverse.

In searching for pertinent techniques, it was suggested that I have a look at *Chisenbop*. Chisenbop is a means of calculating using fingers like an abacus. It is Korean in origin and makes counting on fingers quite a sophisticated art. The right hand covers units 0 to 9 and the left hand covers 10s 0 to 90. The thumbs act as the 5 or 50 carry-over (see next page).

A great tutorial site for this is found on the internet at <http://klington.cs.iupui.edu/~aharris/chis/chis.html>. Any search for Chisenbop will come up with this site. (Some other sites can be found by searching for 'Chisanbop'.)



The Chisenbop 'abacus'.

Maths 'tricks' can be exciting for students and, with the right guidance, have the potential to lead them into all sorts of mind developing activities. There are just so many more fascinating calculation techniques out there that do not involve calculators, all able to enhance your students' arithmetic skills and make them far better users of technology — when they have to be.

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- Kolpas, Sydney J. (2002). Let your fingers do the multiplying. *The Mathematics Teacher*, 95(4), 246–251.

Useful websites

- <http://vedicmaths.org>
- <http://klington.cs.iupui.edu/~aharris/chis/chis.html>
- <http://home.ntelos.net/~clayford/>
- http://www.iit.edu/~sandest/digital_numerics.htm
- <http://www.aare.edu.au/99pap/hei99610.htm>
- <http://www.spatial.maine.edu/~schroedr/digicomp/digicomp.html>
- http://www.writingonhands.org/major_themes/theme4/subtheme_manipulating_time/

The gap between assumed skills and reality in mathematics learning*

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This paper describes the work of the Mathematics Access Centre (MAC) at QUT and the extent of both its effects on student learning and its contribution to staff knowledge of the mathematical difficulties that the diversity of students bring with them to tertiary study. Many of these difficulties can be traced back through schooling and do not become apparent until students need to build on assumed skills in moving into more multi-layered modelling and more extensive applications. A number of changes in the past decade in community and educational beliefs and attitudes have significantly increased the range and diversity of these problems. These will be outlined and discussed with reference to the MAC experience.

Introduction

The past decade has seen an extraordinary range of factors affecting mathematics education in a variety of ways. Acknowledgment of the universal importance of mathematics and of improving access to mathematical confidence for students who traditionally did not 'like mathematics', has led to focus at primary and early secondary level on broadening mathematical activities, and on minimum benchmarks and numeracy. In senior school levels, emphases on contexts, applications, technology and communication, combined with coping with the results of changes in primary and junior secondary mathematics, have increased both opportunities and pressures for teachers. Across the community, mistaken beliefs that increasing computing technology decreases the need for mathematical skills, and lack of understanding of the generic skills that come from mathematics, have provided too many opportunities to de-emphasize or even avoid mathematics, and for spreading misguided beliefs that mathematics is not 'as important' as it used to be. At the tertiary level, the cumulative effects of the above have combined with broader access to tertiary study and economic and conflicting professional pressures on universities, to give a great diversity of students, student preparedness and motivation, and pressure on courses.

* This paper has been subject to peer review.

There is increasing concern at tertiary level across many courses, with decreasing mathematical confidence and the specific or generic mathematical skills that underpin learning in many disciplines. In the UK, a recent survey (Lawson, Croft & Halpin, 2001) indicates that approximately 40% of UK universities have some kind of Mathematics Support Centre provision, and studies (Sutherland and Dewhurst, 1999, Savage and Hawkes, 2000) draw attention to the significant and measurable decline over the last decade in students' basic mathematical skills, even among those with good A-level grades in mathematics. Similar reports and studies in Australia and elsewhere agree, and there is considerable preoccupation with mathematics in engineering (e.g. Coutis, Cuthbert & MacGillivray, 2002) because of both the increasing diversity in mathematical preparedness of students, officially with the same background, and the pressure and lack of flexibility in engineering courses to permit sufficient time for 'catch-up'.

There are three aspects of mathematics that should be considered in any course or discipline: the amount of specific skills needed; the level of generic skills needed; and the time pressures. One reason that engineering mathematics is always under intense scrutiny and pressure is that all three of the above aspects are of the utmost importance in engineering. The extraordinary range of specific skills needed tends to obscure the equally important need for mathematical generic skills, and both are needed within engineering subjects sooner than in any other course. The diversity of engineering students adds another complication. University mathematics support centres (Croft, 2000) all have a significant component in their programs for helping engineering students.

Staff in many disciplines tend to assume specific mathematical skills without checking where or whether they occur now in schooling. There is almost no attention or recognition in other disciplines or courses as to how students acquire the generic skills of mathematics, although there are strong but vague notions that mathematical thinking is important. There are also many mistaken beliefs that mathematical skills and their associated thinking can be acquired very quickly. As students come from secondary school or alternative pathways to university, they arrive with varied expectations and aspirations. Often these expectations and aspirations are grounded in misconceptions and unrealistic ideas gathered from their community, workplace, or high school experience. For many, the need for mathematical confidence and, in tertiary mathematics, mathematical skills, comes as a shock to them. This paper discusses the impact, effects and experiences of a recently established mathematics support centre at the Queensland University of Technology, including qualitative and quantitative evidence of its effects. Its initial programs focussed on engineering mathematics and a variety of first year mathematics subjects. The lessons from these are valuable in themselves and have important implications both for future work in these areas and across a broader educational spectrum.

How the MAC started and why

About 40% of entrants to engineering do not come straight from school. Some have been in the work force for many years, others come through the TAFE sector and others from various bridging programs. Those from high school come with a varied

mathematical background as well; about half have done both Maths B and Maths C (Additional Mathematics at Senior level) and others with only Maths B. As most Queensland mathematics teachers will know, there is a big range of ability between students who have a bottom Sound Achievement and those with a Very High Achievement level in mathematics. The continuum is wide, and similar continuums exist in other courses, including mathematics majors, although to a lesser extent.

Thus, at one end of the continuum, there are students who come to tertiary study with only a few gaps in their knowledge, skills and abilities, and usually can quickly bridge this gap by themselves, provided they have the ability and the motivation to do so. Other capable students come with lots of gaps because of their background and need assistance to bridge these gaps quickly, so that their mathematics learning is not hindered. At the other end of the continuum are not so capable students who have officially been recognised as having the assumed knowledge but often have ‘gaping holes’ in their knowledge, skills and even basic numeracy. This situation has not been helped by the universities softening their entrance requirements to courses for various reasons. Some students, whose course of study requires some mathematics, often do not complete the course because of their struggles with mathematics.

The Mathematics Access Centre (MAC) came about as a direct result of this great diversity of student preparedness and needs. Students come with a wide range of social backgrounds, academic preparedness, learning styles, mathematical skills and expectations of what and how tertiary learning should take place (Coutis, Cuthbert & MacGillivray, 2002). In recognition of these factors, QUT provided a Large Teaching and Learning Development Grant for 2001–2002 to establish the MAC.

The mission of the MAC is to increase access to confidence and achievement in mathematics and engineering mathematics units at first year level. The learning support offered by the MAC is free to students and is taken up voluntarily. It is our contention that the most appropriate and flexible method for bridging the gap between actual and assumed operational knowledge and skills is face-to-face, small-group teaching in an environment of close interaction between students and teachers, as well as between students and students.

The operation of the MAC

The MAC’s suite of programs include enabling tutorials, exam workshops, a drop-in-centre, online registration, diagnostic testing, and student feedback mechanisms. Careful data collection is also carried out, and data systems are being developed, with associated data analysis which provides invaluable insight and evidence for information for staff and authorities, and for ongoing improvements in the MAC programs.

Enabling tutorials

Each week enabling tutorials are offered to students for each first year engineering mathematics unit/subject, and for first year mathematics units/subjects, with the latter sometimes grouped in pairs of units/subjects with similar cohorts. These tutorials are student-driven, and encourage students to explore their understanding and skills in a

friendly, informal, openly frank environment. These tutorials have numbers ranging from 5–20 and the staff member will start on problems that are similar to those given in the formal classes at that time, to provide a starting point for motivation and questions. Students are encouraged to ask about whatever they want, to interact as much as possible with their fellow participants, and not to be shy or reluctant to ask about any matter. The staff for these tutorials are all experienced teaching academics in mathematics (or statistics as appropriate) who have the skills to identify and help address weaknesses in foundation skills: correcting inappropriate learning approaches, improving confidence and generally providing support and encouragement for students anxious about their mathematical ability and background.

The students who attend these tutorials are vocal in their online and direct feedback, in their praise of the help received in gaining confidence in their mathematics.

Exam workshops in engineering mathematics

A few days before summative assessment is given in engineering mathematics, the MAC provides an exam workshop. These workshops are typically two hours in duration, but for a final exam there are often up to three sessions on different parts of the exam, allowing students to choose a session for particular help. These workshops are designed to consolidate and synthesise skills acquired in enabling tutorials and mainstream teaching. The teaching strategies focus on helping students to develop problem-solving techniques and mathematical thinking using problems similar in level and style to previous examination questions. These workshops are very popular with students.

Drop-in centre

The MAC operates a Mathematics Drop-in Centre from Monday to Friday, which is open from 8 am to 7 pm. Students can use this room for quiet study, as a place to work in groups, and as a place to get help with mathematics problems. Help is available for four or five hours each day from volunteer academic staff and paid senior students. During the middle of the day (11 am – 3 pm) there tend to be between ten and twenty students there at any one time. At other times, the attendance varies from none to ten. Not only do students appreciate the informal assistance on-call, but the combination of the drop-in centre and the MAC tutorials and workshops has promoted an environment of fellowship in learning.

Successes and challenges

The effects of the MAC can be measured qualitatively through online and direct student feedback, as quoted in above comments, quantitatively through statistical analysis of MAC and student data, and indirectly through effects carried over to second year and beyond. All of these indicate that the MAC is highly successful, but there are also a range of challenges.

Successes

Positive qualitative feedback is indicated above; there is almost no negative feedback either anonymously or directly. Statistical analysis to date focusses mostly on the first year engineering mathematics subjects, but, based on initial experience, systems have been put into place to obtain data of reasonable quality to monitor usage of the drop-in centre and other aspects of the MAC.

The statistical analysis of first year engineering mathematics takes account of students' school subjects, their first piece of assessment each semester, and, in their second engineering mathematics subject, the result in their first engineering mathematics subject. For any student, usage of the MAC workshops or tutorials makes a statistically significant difference in performance, but closer statistical examination is even more revealing. Students entering with the senior extension subject (Maths C in Queensland) do a different subject in their first semester to those with just the core senior maths subject (Maths B in Queensland), but the second semester subject is the same. First note that attendance at exam workshops is highly correlated with enabling tutorial attendance in all three subjects. In their first semester, for those with Maths C, attendance at the exam workshops has more effect than tutorial attendance, but within those students who attend at least some segment of one of these, the amount of time spent at either workshops or tutorials is not significant. However for those students with just Maths B, not only are both workshop and tutorial attendance significant, but within the group who attend at least some segment, the amount of time spent at enabling tutorials is significantly beneficial. In the second semester subject, after allowing for first semester result, first piece of assessment result and workshop attendance (all three being statistically significant and beneficial), their school background is still highly significant, on average giving a difference of 10% in the final mark after allowing for the other variables.

The above analysis indicates that those students with the extension school mathematics can succeed even if not particularly capable, provided they engage. Those without the extension school mathematics need to engage and they benefit significantly from extra face-to-face help, but the advantages of the extension school mathematics subject are long-lasting.

The year after the establishment of the MAC, there was a significant drop in failure rates in second year mathematics subjects for engineers. This indicated that the MAC not only helped students in their first year, but also helped students acquire sufficient confidence and learning skills to take with them into subsequent study.

Challenges

Both the quantitative and qualitative evidence demonstrate that students who use the MAC tutorial/workshop programs improve their performance and confidence. Data for the drop-in centre do not track individual students, just the usage by course, subject, day and time. Although engineering students are using the MAC programs in a significant way, there are still many who do not, but should. It must be emphasised that the MAC is voluntary and is additional to a well-organised and supportive subject program, so is not intended for everyone; however, between 70% and 90% of those who

fail did not use the MAC, while between 75% and 90% of those who passed did use the MAC. Using the MAC could be interpreted as a measure of engagement on the student's part, but evidence indicates that the MAC helps the students to engage as well as helping them gain in self-knowledge and confidence.

It is apparent that different MAC strategies are needed by first year students taking mathematics major subjects. The challenge is to increase the use of the MAC by those students who are struggling with the work and 'hiding' from help. Once the mathematics majors students start coming, they keep coming, and greatly appreciate the atmosphere of collegiality and informal, student-driven face-to-face help. Some progress has been made by grouping mathematics major subjects together and identifying 'starter' topics each week with particular staff.

Typical student difficulties

Apart from a (currently) small number of students entering without official assumed knowledge, for whom, with extraordinary commitment and much help, success is possible rather than probable, we are considering students who have passed Year 12 core mathematics, and who choose mathematically-based courses. Although there is great diversity, difficulties can be generally described in terms of lack of algebraic thinking and lack of systematic approaches. The specific areas of difficulty tend to originate in middle school, and it seems that if capable students do not have sufficient opportunity to attain the mathematical skills at the stages when they are ready for them, the consequent weaknesses tend to persist. Even the most outstanding students are not immune to such weaknesses, but such students can strengthen themselves with a little assistance.

For the less capable (by tertiary standards) or those with more extensive weaknesses, the following comments from an experienced school and tertiary teacher illustrate the difficulties:

The problems occur when a 'basic' is a tiny part of a larger problem... Because the 'basic' is not second nature they rush it or confuse it and hence get it wrong. I've seen this time and time again. For example, engineering students... with problems... will often know how to proceed in a given complex problem but mess up a 'basic' and as a result get to a point where they can proceed no further. Even if we could find the time to devote to 'basics' there is the question of the morale of the students who are weak in 'basics'. I have a feeling that this is a reason that many who need help do not seek it. They feel foolish because they cannot do things that they see as 'simple'. (Carter, 2002)

The following illustrate some typical difficulties that tertiary teachers share with senior mathematics teachers.

Fractions: Algebraic confidence with fractions is essential across all disciplines with at least some mathematical basis or thinking. Difficulties with fractions are very widespread amongst all student types and capabilities, and stem from higher primary level.

Factoring: A range of factoring problems from number skill difficulties to algebraic factoring are common; an easy confidence with brackets is essential for all mathematical modelling and thinking.

Sketching curves: One of the strengths of good mathematical thinking is having a feel for a function and its associated graph. Some students cannot visualise a straight line graph from a linear function. They have difficulty seeing how changing one of the parameters will effect the graph, and seeing the pattern in functions and functional behaviour.

Trigonometry: Contrary to popular general belief in middle schooling, trigonometric functions appear in many and varied disciplines. Again, it is the understanding of the functional behaviour of trigonometric functions that is vital, as it is these functions that form the basis of modelling periodic behaviour.

General algebraic skills: It can be seen that the above are components of general algebraic and functional skills. Such skills underpin all representational and relationship thinking and modelling: in industry, finance, information technology, health and life sciences, social sciences, as well as engineering and physical sciences. When the students themselves say they do not have enough algebraic skills, it is time to listen to them.

Transferability: Transferring knowledge and skills from one context to another is an essential characteristic of mathematical training, and vitally important for mathematics majors and engineering students. Transferability involves algebraic confidence, pattern recognition (no matter what notation is used), and knowledge of one's own mathematical 'toolbox'.

Students often lack confidence to attempt some mathematical thinking and writing in case it is wrong. The MAC 'enabling tutorials' reinforce the emphasis of the usual tutorials: to 'have a go', and to make trying a learning experience in a collegiate atmosphere. Those with some confidence in an area will have a 'toolbox' of 'maths' that allows them to make good progress in more complex problems. Many students have a 'toolbox' but have not been made aware of what is in it or how to use it. The MAC allows time for students to 'delve' into their 'toolbox' and discover what is there, and helps students to recognise that the work put into mathematics is directly related to output.

The emphasis on minimum standards has had a deleterious effect on the more capable students. This has gone so far that some teachers teach and test to the minimum standard and often forget to challenge the more capable students in teaching and testing.

'Just passing' at Year 12 mathematics to enter university mathematics/engineering courses soon catches up with capable and not so capable students when all the little things they were able to avoid are suddenly needed to solve problems: such things as completing the square, basic trigonometry definitions and standard formulae, sketching of simple graphs, factorisation identities, etc.

Conclusion

Mathematics is as fundamental to, and interwoven with, human activities, achievements and endeavours, as language. Mathematics education is susceptible to excessive focus on selected components at any one point or educational level. Over the past decade, the focus on improving access to mathematics for all students and on applicability and investigations has been both needed and valuable, but the interests of the top half of the mathematics capability continuum have not been sufficiently protected. Integrating depth and breadth of mathematics learning in investigations requires significant mathematical and teaching expertise and commitment. Such aspects as a general lack of recognition of the generic skills that come from doing mathematics, over-prolonged emphasis on spatial and measurement aspects at the expense of algebraic thinking, and lack of understanding that increased technology increases the need for mathematical skills, have combined with insufficient numbers of teachers with mathematical comfort, confidence, and qualifications.

The type of student-driven, collegiate, face-to-face support from expert and experienced staff that the MAC provides is always valuable; in the current situation it is invaluable for students and for the information that such programs can feed back into the system to improve the educational framework for all students.

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How to construct the locus of a point which satisfies the focus and eccentricity definitions of a conic section by using the TI-92 plus*

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In analytic geometry, there are two ways to define the locus of a conic section, called the focus definition and the eccentricity definition. When the students study these definitions, they cannot imagine the locus of a point which satisfies these definitions, especially the eccentricity definition. Without any technology, it can be very difficult for teachers to explain to their students how the locus of that point can be obtained.

This presentation shows how to use the TI-92 Plus graphing calculator to obtain the locus of a point which satisfies each of the two definitions of a conic section. With the help of this tool, students are able to have a better grasp of the definitions of a conic section.

Introduction

There are several definitions of a conic section. The first definition is called the *focus definition*, which can be stated as follows.

Definition 1 An *ellipse* is the locus of all points such that the sum of the distances between these points and two fixed points is constant and greater than the distance between the two fixed points.

Definition 2 A *hyperbola* is the locus of all points such that the difference of the distances between these points and two fixed points is constant.

The second definition of a conic section is called the *eccentricity definition* or the *focus-directrix definition*, which can be stated as follows.

Definition 3 A conic section is the locus of all points such that the ratio of the distances between these points and a fixed point to the distances between these points and a fixed line is constant. The ratio is called an *eccentricity* and is denoted by e .

* This paper has been subject to peer review.

- If $e = 1$, then the locus is a parabola.
- If $e < 1$, then the locus is an ellipse.
- If $e > 1$, then the locus is a hyperbola.

The locus of a point which satisfies the focus definition

1. Ellipse

The following processes show you how to construct the locus of a point which satisfies definition 1.

1. Take two fixed points (foci), say A and B. Construct a segment \overline{CD} and take a point E on \overline{CD} and then construct two segments \overline{CE} and \overline{DE} , as shown in Figure 1. Draw two circles, the first circle has radius CE with its centre at A, and the second circle has radius DE with its centre at B. Take two intersection points of the two circles, call P and Q (see Figure 2).

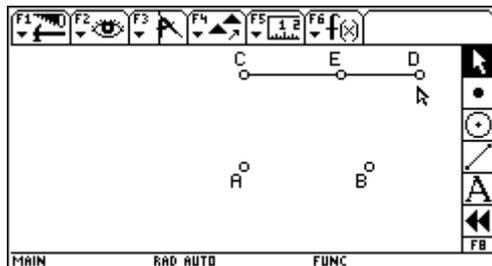


Figure 1

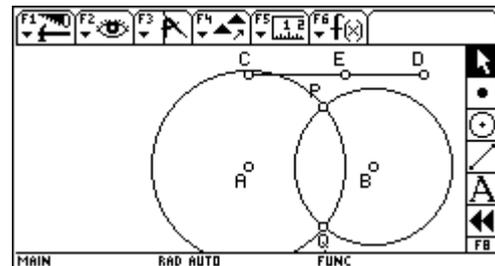


Figure 2

2. Draw four segments \overline{AP} , \overline{AQ} , \overline{BP} and \overline{BQ} . We can see that

$$AP = AQ = CE \text{ and } BP = BQ = DE$$

Then $AP + BP = CE + DE = CD$ and $AQ + BQ = CE + DE = CD$. We can show these facts by using the length command in the measure tool. Animate the point E along \overline{CD} ; we can see that these facts are still and always true, as shown in Figure 3. These mean that the points P and Q satisfy definition 1. Then the locus of P and Q must be an ellipse.

Hide both circles, trace the points P and Q and animate the point E along \overline{CD} . Then we obtain the completed ellipse (see Figure 4).

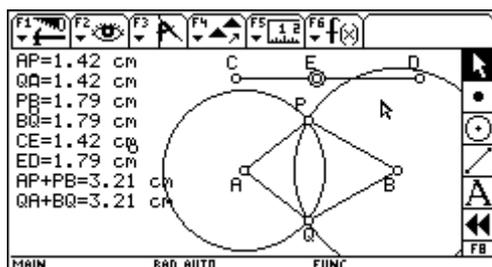


Figure 3

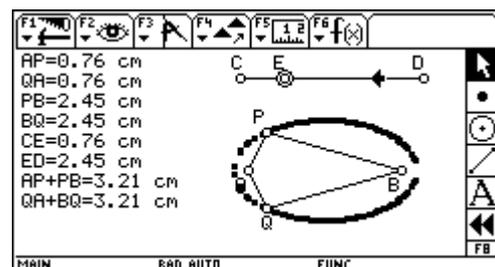


Figure 4

2. Hyperbola

The processes to obtain a locus of a point which satisfies definition 2 are similar to the locus of a point which satisfies the definition 1, but it is a little more complicated as the following shows.

1. Take two fixed points A and B. Construct \overline{CD} and take a point E on \overline{CD} . Construct \overline{DE} and take a point F on \overline{DE} . Construct \overline{CF} , \overline{EF} and \overline{CE} . Measure CF, EF, $CF - EF$ and CE. We can see that the $CF - EF$ is constant and equal to CE. This is always true when we animate the point F along \overline{DE} (see Figure 5).

Draw two circles C_1 and C_2 , where C_1 has its centre at A and radius equal to CF, C_2 has its centre at B and radius equal to EF. Construct the two intersection points of C_1 and C_2 , call G and H (see Figure 6). Since $AG = CF$, $BG = EF$, $AG - BG$ must be constant and equal to CE. This shows that the points G and H satisfy definition 2 of the hyperbola. But the locus of G and H just only be the right half side of the hyperbola.

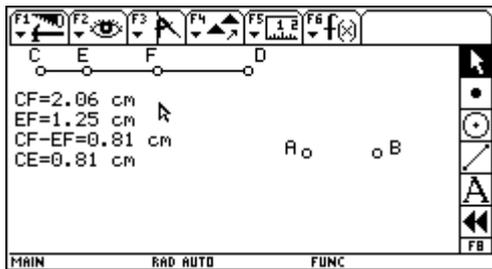


Figure 5

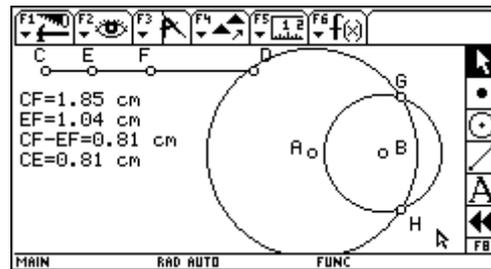


Figure 6

2. Now, we will construct the left half side of the hyperbola. Hide the circles C_1 and C_2 . Construct another two circles C_3 and C_4 , where C_3 has its centre at A and radius equal to EF, C_4 has its centre at B and radius equal to CF. Construct the two intersection points of C_3 and C_4 , and label them I and J (see Figure 7). The points J and I also satisfy definition 2.

Hide the circles C_3 and C_4 . Trace the four points G, H, I and J and animate the point F along \overline{DE} . We get a completed hyperbola (see Figure 8).

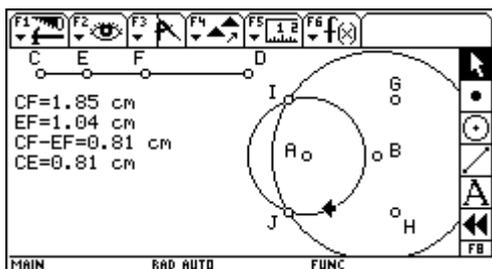


Figure 7

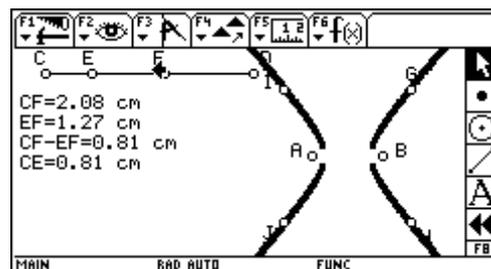


Figure 8

The locus of a point which satisfies the eccentricity definition

Take a fixed point F as a focus and take a fixed line L as a directrix. Consider the locus of a point P such that the ratio of the distance between P and F to the distance between P and the line L is constant. Suppose that

$$\frac{PF}{PQ} = e$$

where Q is the point on L such that \overline{PQ} is perpendicular to L .

1. Parabola

The locus of P is a parabola if $e = 1$ or if $PF = PQ$. The following processes show one method to construct the locus of a parabola.

1. Draw a fixed line L as the directrix and a fixed point F outside L as the focus of a parabola. Construct a point Q on L and draw a segment \overline{QF} (see Figure 9).

Draw a line through Q and perpendicular to L , calling it L_1 . Construct the midpoint of \overline{QF} , calling it A . Draw a line through A and perpendicular to \overline{QF} , calling it L_2 . Construct the intersection point of the lines L_1 and L_2 , calling it P (see figure 10).

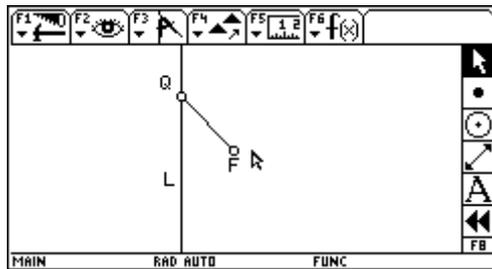


Figure 9

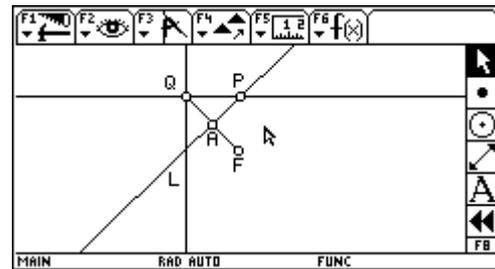


Figure 10

- Construct \overline{PQ} and \overline{PF} and hide the lines L_1 and L_2 . Measure the length of \overline{PQ} and \overline{PF} . We can see that $PQ = PF$. This result is always true when we move the point Q along the line L (see Figure 11). This means that the point P satisfies definition 3 with $e = 1$. Then the locus of P must be a parabola.

Trace the point P and animate the point Q along the line L . Then we obtain the completed parabola, as shown in Figure 12.

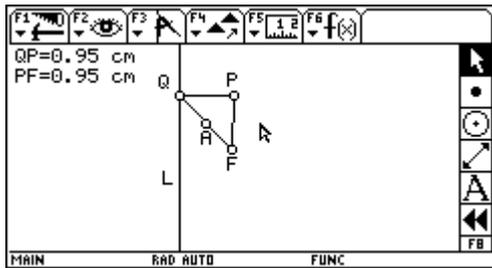


Figure 11

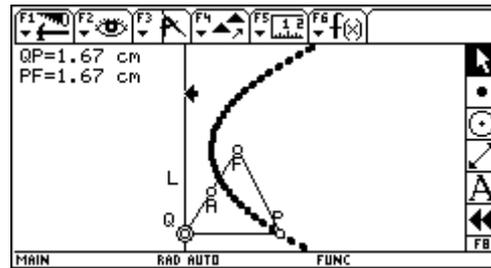


Figure 12

2. Ellipse

- Draw a segment \overline{AB} and a ray from B . Take a point C on the ray and construct \overline{AC} . Take a point D on \overline{AB} ; draw a line through D and parallel to the ray. Construct the intersection point of this line and \overline{AC} , calling it E (see Figure 13).

Hide the line through D and E , and construct \overline{DE} and \overline{BC} . Measure the length of \overline{DE} and \overline{BC} and find the ratio $\frac{DE}{BC}$. When the point C moves along the ray, we can see that this ratio is constant and less than 1 (see Figure 14). We will use this idea to construct the locus of an ellipse.

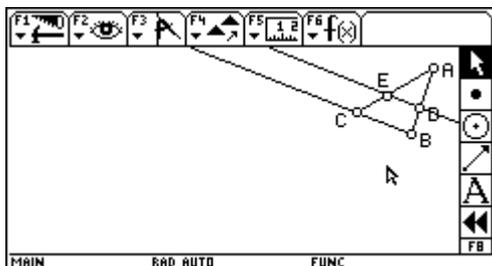


Figure 13

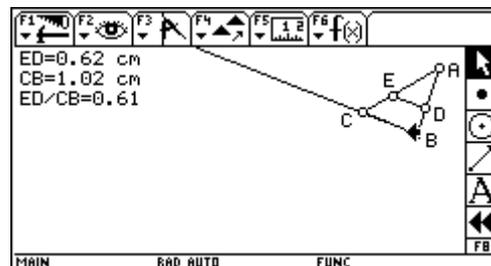


Figure 14

- Draw a fixed line L as a directrix and a fixed point F outside L as a focus of an ellipse. Construct a line passing through the point F and perpendicular to L . Draw the first circle with its centre at F and radius equal to DE . Draw the second circle

with its centre at F and radius equal to BC. Construct the intersection point of the second circle and the line through F, calling it G (see Figure 15).

Hide the second circle. Construct the third circle with its centre at G and radius equal to DE. Move the point F until the third circle intersects the directrix L at two points, calling them H and I (see Figure 16).

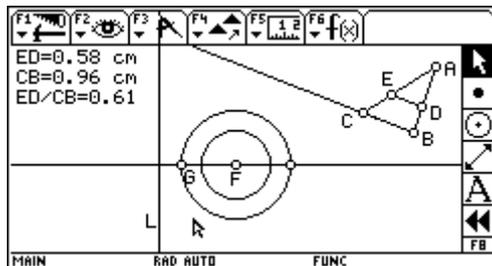


Figure 15

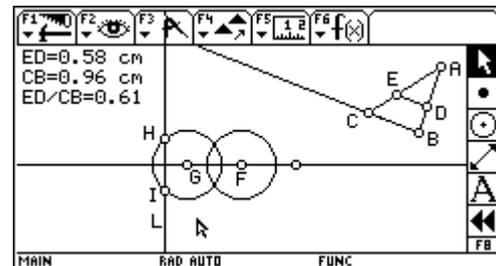


Figure 16

- Hide the third circle. Construct the fourth circle with its centre at H and radius equal to BC. Take the upper intersection point of the fourth and the first circles, calling it P. Hide the fourth circle. Construct the fifth circle with its centre at I and radius equal to BC. Take the lower intersection point of the fifth and the first circles, calling it Q. Hide the fifth circle (see Figure 17).

We can see that $PH = QI = BC$ and $PF = QF = DE$.

Therefore the ratios $\frac{PF}{PH} = \frac{QF}{QI} = \frac{DE}{BC}$ must be constant and less than 1.

This shows that the points P and Q satisfy the eccentricity-definition with the eccentricity less than 1. Then the locus of P is the upper half of an ellipse and the locus of Q is the lower half of the ellipse.

Trace the points P and Q, and animate the point C along the ray. Then we get the completed ellipse (see figure 18).

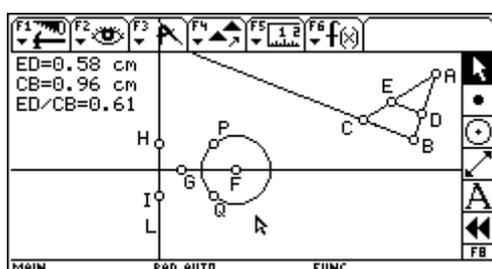


Figure 17

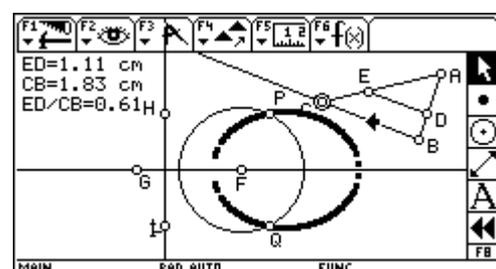


Figure 18

3. Hyperbola

The process of obtaining the locus of a hyperbola is similar to the process of obtaining the locus of an ellipse; process (1) is particularly similar.

1. Draw a segment \overline{AB} and a ray from B. Take a point C on the ray and construct \overline{AC} . Take a point D on \overline{AB} and draw a line through D and parallel to the ray. Construct the intersection point of this line and \overline{AC} , calling it E (see Figure 19).

Hide the line that through D and construct \overline{DE} and \overline{BC} . Measure the length of \overline{DE} and \overline{BC} and find the ratio $\frac{BC}{DE}$. When the point C moves along the ray, we can see that this ratio is constant and greater than 1 (see Figure 20).

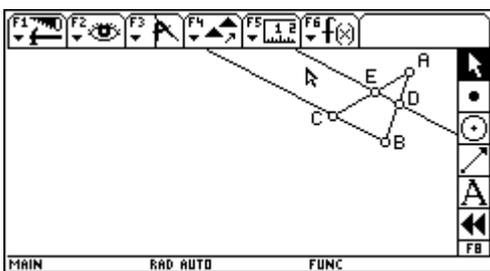


Figure 19

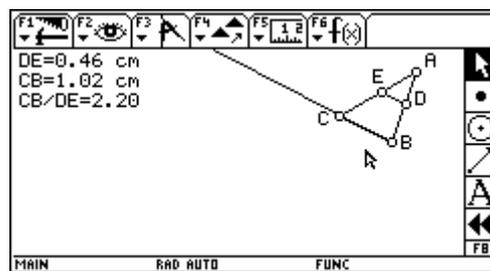


Figure 20

- (2) Draw a fixed line L and a fixed point F outside L. Construct a line passing through the point F and perpendicular to L. Draw the first circle with its centre at F and radius equal to BC. Draw the second circle with its centre at F and radius equal to DE. Construct the intersection points of the second circle and the line through F, calling them G and H (see Figure 21).

Hide the second circle, and construct a third circle with its centre at G and radius equal to BC. Construct the two intersection points of the third circle and the directrix L, calling them I and J. Hide the third circle (see Figure 22).

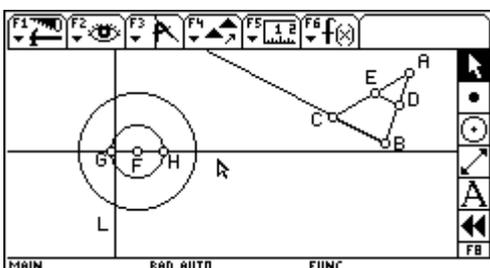


Figure 21

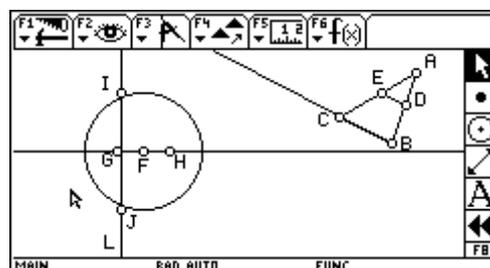


Figure 22

3. Construct the fourth circle with its centre at H and radius equal to BC. Construct the two intersection points of the fourth circle and the directrix L, calling them K and M. Hide the fourth circle (see Figure 23).

Draw a sixth circle with its centre at I and radius equal to DE; construct the right intersection point of the first and the sixth circle, calling it N. Hide the sixth circle.

Draw a seventh circle with its centre at J and radius equal to DE; construct the right intersection point of the first and the seventh circle, calling it O. Hide the seventh circle (see Figure 24).

We can see that $FN = FO = BC$ and $IN = OJ = DE$.

Then the ratios $\frac{FN}{IN} = \frac{FO}{OJ} = \frac{BC}{DE}$ must be constant and greater than 1.

So, the points N and O satisfy the eccentricity-definition with $e > 1$. The locus of N is the upper right hand side of a hyperbola and the locus of O is a lower right hand side of a hyperbola.

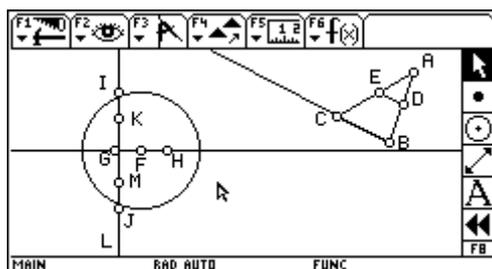


Figure 23

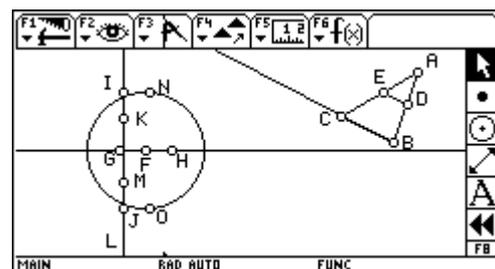


Figure 24

4. Draw the eighth circle with its centre at K and radius equal to DE. Construct the left intersection point of the first and the eighth circle, calling it P. Hide the eighth circle.

Draw a ninth circle with its centre at M and radius equal to DE; construct the left intersection point of the first and the ninth circle, calling it Q. Hide the ninth circle (see Figure 25).

Similarly, the ratios $\frac{PF}{PK} = \frac{QF}{QM} = \frac{BC}{DE}$ are constant and greater than 1. The locus of the points P and Q are respectively the upper left hand side and the lower left hand side of the same hyperbola.

Trace the four points N, O, P and Q and animate the point C along the ray. We obtain the completed hyperbola (see Figure 26).

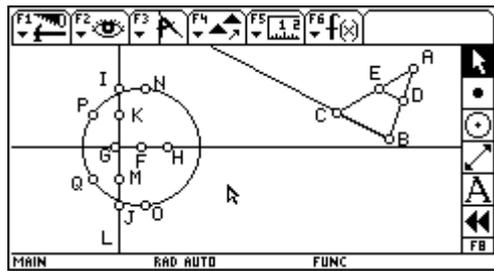


Figure 25

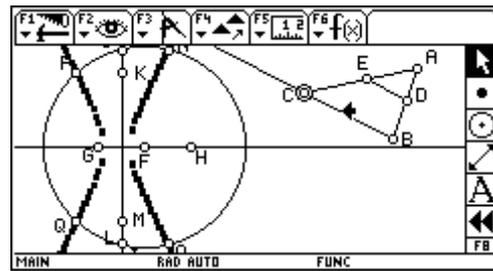


Figure 26

Conclusion

By taking advantage of the capabilities of the TI-92 Plus, teachers have an option of presenting ways of constructing the locus of a point which satisfies the definition of a conic section. Students then have ample opportunities to explore — with critical reasoning — as to why the two definitions end up with the same result. Teachers should encourage their students to discuss what has happened at each step of the construction or offer some thought provoking questions to motivate them to discover the next step by themselves.

Teachers' and students' beliefs about the benefits of computer use for secondary mathematics learning^{2*}

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Society appears to have great faith in the educational value of computers. It is widely believed that computer use promotes learning. Mathematics curriculum documents are replete with statements advocating the beneficial outcomes of computer use. The Victorian Mathematics CSFII has an accompanying ICT chart, and teachers are advised that computer applications are integral to students' learning experiences. Based on survey and observational data from a study in which equity issues and perceptions of computer use for secondary mathematics learning are being explored, the findings on teachers' and students' beliefs about the benefits of computer use will be presented and discussed.

Introduction

In Australian society today, there would appear to be much faith in what computers can do to improve many aspects of our daily lives. Beliefs seem strong that computer use will also improve educational outcomes. An examination of curriculum documents in Australia and in other, mainly Western, nations reveals a plethora of statements about incorporating computers into mathematics teaching/learning. The impression gained is that computers will benefit students' learning of mathematics. Consider, for example, the following general statement drawn from the *Overview of the Victorian Curriculum and Standards Framework II – online* (Victorian Curriculum and Assessment Authority, 2001a):

The CSF encourages full use of the flexibility and value for teaching and learning programs provided by the increased application of information and communications technology (ICT).

The CSF acknowledges that through the effective use and integration of ICT students are quickly developing new capabilities and that teachers have greater

² This study was funded by the Australian Research Council. The author's thanks are extended to Nike Prince who assisted in the data gathering and analyses.

* This paper has been subject to peer review.

choice in creative teaching, assessment techniques and connections to students learning at home.

An *Information and Communications Technology [ICT] chart* (Victorian Curriculum and Assessment Authority, 2001b) accompanies the *Mathematics CSFII – online*. The ICT chart reveals that it is expected that students at Levels 5 and 6 (grades 7–10) will use the following ICT applications and skills: file management, graphics, multimedia, electronic communication, data-logging, database, spreadsheet, desktop publishing, simulation/modelling, graphics calculators and computer algebra systems [CAS].

To what extent, does the reality in mathematics classrooms match the rhetoric of these proclamations and expectations? Do secondary teachers of mathematics and their students use computers for learning mathematics and do they believe that computer use affects the students' mathematics learning outcomes? These questions are being explored in an on-going research study on equity and the use of computers for the teaching and learning of secondary mathematics. As part of the study, data were gathered on teachers' and students' beliefs about the benefits associated with computer use for learning mathematics. The data sources included: large and small scale survey data, classroom observations, interviews with teachers and students, and students' post-lesson reflections. The findings are reported in this paper.

The study

There are three major stages of data collection associated with the study.

Stage 1

In 2001, surveys were administered to grade 7–10 mathematics teachers and to their students in a representative sample of 29 schools across Victoria. To preclude school effects in the results, only one class at each grade level in each school was involved. The total sample size was as follows:

Grade 7–10 teachers: 96 (M = 44; F = 52)

Grade 7–10 students (Grade 7–10): 2140 (M = 1111; F = 1015; unknown = 14).

Grade 11 students were also surveyed but the results are not discussed in this paper.

Stage 2

In three schools, data were gathered from mathematics co-ordinators, and from the teachers and students in two grade 10 mathematics classes. The mathematics co-ordinators provided background information on the mathematics programs, the number and configurations of computers in the school, and statistical data on Victorian Certificate of Education [VCE] mathematics enrolments and success rates over the last few years. The grade 10 teachers completed background information sheets on the students in their classes, completed the same survey instrument as was used with teachers in 2001, were interviewed, and their classes were observed for two one-week periods about a month apart; the lessons in the second week were videotaped. The students completed the same survey instrument as was used with students in 2001 and

filled out *Today's lesson* reflection sheets at the end of each observed lesson. Four students in each class (2M, 2F) were also interviewed.

Stage 3

In 2003, the large scale surveys will again be administered to teachers and students in the same schools as in 2001.

Data gathering

Items of interest

Included in the teacher and student survey questionnaires were the following questions:

1. Do you believe that computers help people understand mathematics?
Yes / No / Unsure
Why do you say this?
2. Do you believe that computers help your students (you) understand mathematics?
Yes / No / Unsure
Why do you say this?

Similar questions were included in the *Today's lesson* sheets and were also asked of the teachers and students in the interviews. Data from the whole sample of teachers and students surveyed in 2001, and from the two teachers and four students at one of the three schools involved in 2002 are presented and discussed below.³

³ Data from the other two schools had not been fully analysed at the time of writing this paper.

Results and discussion

Stage 1 results

The percentage of responses by category (Yes / No / Unsure) to Questions 1 and 2 (above) of the 96 teachers and the 2140 students are shown in Figures 1 and 2.

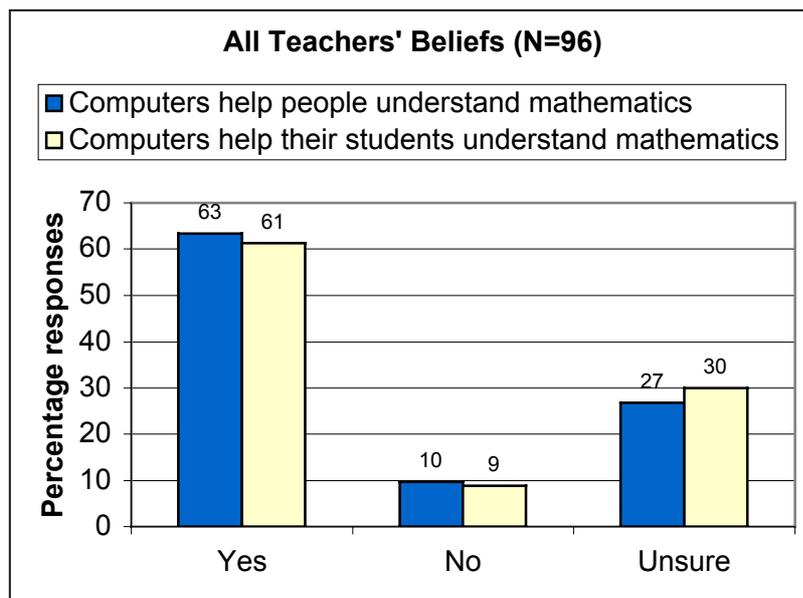


Figure 1. Teachers' responses

It can be seen from Figure 1 that 63% of the teachers believed that computers help people understand mathematics and 61% that their students' understanding is helped by computer use; about 10% of the teachers disagreed with both of these questions. In response to each question, a fairly sizeable proportion (over 25%) was uncertain if computers helped in the understanding of mathematics.

Among the students, the pattern of beliefs was quite different (see Figure 2). Fewer than 30% agreed that computers either helped people generally, or themselves in particular, to understand mathematics. Forty-one percent believed that computers did not help them understand mathematics but only 29% believed that they did not help in the mathematics understanding of people in general. Large proportions of students were uncertain one way or the other (42% about people generally, and 33% about themselves).

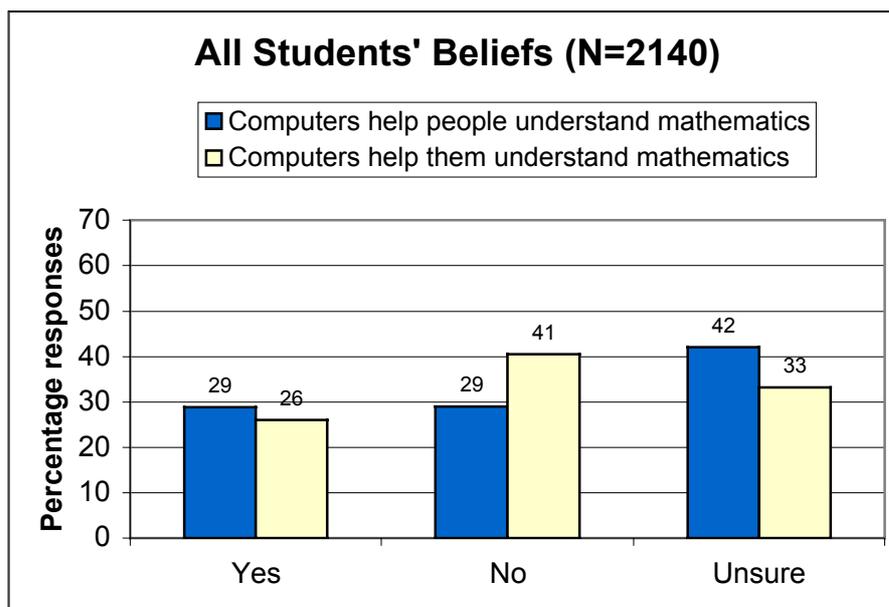


Figure 2. Student's responses

The explanations for their responses to the question about computers helping students understand mathematics provided by teachers and students were enlightening. Representative examples of the explanations in the affirmative and in the negative are shown in Table 1.

In Table 1, it can be seen that enjoyment and the speed at which computers display 'solutions' were common reasons given by teachers and students who believed that computers helped students in their understanding of mathematics. Teachers who did not believe computers helped their students' understanding tended to be somewhat cynical and negative about computers and about students' behaviour with the computers (see Table 1). The students appeared to be more perceptive in their reasons for not believing that the computers helped their understanding.

Table 1. Reasons given by teachers and students for their beliefs that computers do/do not help students' understanding of mathematics

Do you believe that computers help your students (you) understand mathematics? YES	
Teachers' explanations	Students' explanations
<ul style="list-style-type: none"> • sometimes gives them a different way/ perspective on the mathematics involved • enjoyment, different to just doing paper calculations, produce accurate good looking graphs etc. — make predictions • particular software saves time and verifies their understanding. Computers allow them to carry out problems / exercises / questions quicker • students have different learning styles and many are very familiar with using a computer. This can then be used as a tool for learning mathematics 	<ul style="list-style-type: none"> • because it's learning in a fun way • because sometimes computers explain things better • I understand computers for I spend a lot of time around them and it is not easy for me to get confused • because computers explain more than the teacher does. When the teacher says something the students might forget it
Do you believe that computers help your students (you) understand mathematics? NO	
Teachers' explanations	Students' explanations
<ul style="list-style-type: none"> • it is just an instrument to arouse enthusiasm • the students still see a computer lesson as a 'slack' lesson — or a 'fun' lesson. Because they mostly need to read instructions, they rarely understand exactly what we are trying to get them to master • these students learn just as well without computers 	<ul style="list-style-type: none"> • because it's just as easy without computers • because if you know mathematics its in your head but to do mathematics on the computer you have to know how to do something else • because I do fine without them plus I don't like them • because the computer does everything I don't need to think, and therefore I don't learn • because it's easier to understand when it's on paper in front of you

Stage 2 results

The results from only one grade 10 class of 25 students (11F & 14M), many of whom were NESB, attending a large, outer metropolitan, government school in a low SES area are presented here. The teacher and students completed the same survey questionnaires used in 2001 with the large samples of teachers and students. The grade 10 mathematics teacher provided background information about the students. Lessons in which computers were used for mathematics learning were observed over two one-week periods. At the end of each of these lessons, the students completed *Today's lesson* sheets. Four students (2M, 2F) and the mathematics teacher were also interviewed.

Students' responses

The 25 students' responses to the question on the survey about whether computers helped them understand mathematics are shown in Figure 3. Also shown are the percentage responses to a similar question – Did using the computer help your understanding of the mathematics? (Yes / No / Unsure. Explain) – that was included on the *Today's lesson* sheets and filled in after each lesson. For these responses, the percentages calculated are based on the mean response to each category over a five-day observation period, during which the students were using the software application, *Geometer's Sketchpad*. During the week, they were working with triangles, constructing perpendicular bisectors of the sides of triangles, angle bisectors, drawing in-circles and circumcircles, constructing tangents to circles, examining 'angles at the centre' and 'angles at the circumference' and the relationship between the two, and constructing cyclic quadrilaterals.

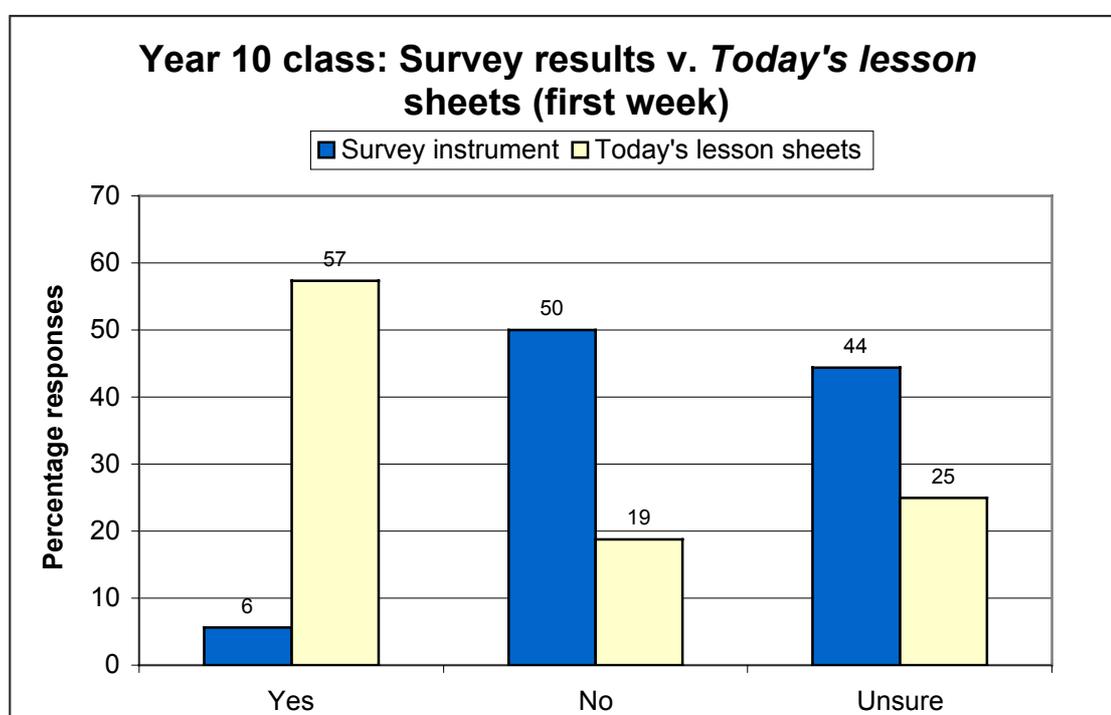


Figure 3. Beliefs about computers helping mathematics understanding: Grade 10 students' responses to survey and *Today's lesson* question.

When the survey responses to the question, 'Do computers help you (students) understand mathematics?' (Question 2 above) of the grade 10 students (Figure 3) are compared to those of the 2140 students surveyed in 2001 (Figure 1), interesting differences were found. The grade 10 class was much more negative in their beliefs: 5% responding 'Yes' and 50% responding 'No', compared to 26% saying 'Yes' and 41% saying 'No' in the large sample.

Of particular interest was the difference in the grade 10 students' response patterns on the survey (completed before the period of classroom observations) and their responses on the *Today's lesson* sheets (see Figure 3). There was a large increase from 5% agreeing on the survey that computers help them understand mathematics to 57% (on average) saying that *Geometer's Sketchpad* had helped their understanding of the geometrical concepts explored.

When asked at interview if computers helped them understand mathematics, the responses of the four students were as follows:

- S1: You get a better understanding and a feel for the shapes. As an example, the tangents — you could move the shape around and see how it is relevant to the shape and how it changes and how [it] alters etc., whereas, if you were doing this by hand without a computer you wouldn't, it wouldn't be so diverse and you wouldn't be able to see.
- S2: It just didn't help me understand, yeah... 'cause with maths I like to do more of the working out and see how it becomes, it's more with equations and stuff than with drawing graphs and circles, yeah.

This student felt that computers were not much help but was not dismissive:

- S3: Not much but they're some use [for] calculating equations, drawing up graphs and geometry.'
- S4: It makes like everything easier to see. [Interviewer: An example?] Like drawing parabolas, it like shows the difference of like what the values do to a graph, like it shows it more clearly so you can like easily spot the difference, whereas if you draw it, it might be squashed together or something.

The students' responses reveal the impact of the visual images that are so easily produced by the software. Some students, however, appear to enjoy the challenge of producing these images themselves. Clearly there is evidence here of the effects of individuals' preferred learning styles.

The teacher's responses

On the survey and at interview the grade 10 mathematics teacher indicated that he believed that using computers helps his students understand mathematics better. The reasons he gave were:

- students (& parents) relate computers to learning;
- it is visual;
- they [students] have to apply concepts.

The teacher seems to be in touch with parental expectations that computers are beneficial for the education of their offspring. He also appears to believe in the power of the visual. The teacher also noted that students of all academic abilities and with all types of learning patterns can benefit from using computers for mathematics learning.

Conclusions

The large scale survey data revealed that grade 7–10 teachers seem more convinced than their students of the benefits of computers for people’s and students’ understanding of mathematics

In the Year 10 class, the proportion of students who believed computers help them understand mathematics was lower than in the large-scale sample. After a week using *Geometer’s Sketchpad* in their mathematics classes, the proportion saying that the computers had helped their understanding of the mathematics had substantially increased. However, there remained a large group that were unconvinced (about 40%). Preferred learning styles may contribute to the explanation. Another question that arises, however, relates to the particular software that was used. Would the same students respond in the same way if a different software package had been used and/or if the computers had been used for the learning of different mathematical concepts?

It appears that the experience of using the computers for mathematics learning — with appropriate software and teacher facilitation — can affect students’ beliefs about the potential of computers to improve their understanding of the mathematics encountered.

The findings from the study reported here only show the potential for change in beliefs or perceptions. More evidence is needed, however, to support society’s faith in computers for improving mathematics learning outcomes.

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'The ends and sides of a topic': How has the use of the CAS calculator affected the teaching and learning of mathematics in the Year 12 classroom?*

Sue Garner

Ballarat Grammar School, Victoria

This paper describes the journey of change in the approaches to teaching and learning that has been experienced at Ballarat Grammar School in Victoria. Ballarat Grammar is one of the three original pilot schools in the VCAA CAS Pilot Project for Units 1–4 Mathematical Methods. This Pilot Project has emanated from the CAS-CAT Project (Computer Algebra Systems: Curricula, Assessment and Teaching Project). Students in Year 12 have had access to the Casio FX 2.0 calculator for over two years and have sat their final exams in November 2002 in the new subject Mathematical Methods CAS. It has been found that there is an explosion of methods in learning that occurs in the CAS classroom and that teaching radically changes in the senior mathematics classroom when students are using the CAS calculator on an everyday basis.

Introduction

Thirty students at Ballarat Grammar School have travelled a journey that has culminated in their Year 12 final mathematics examinations at the end of 2002. These students have learnt the use of the Casio FX 2.0 and FX 2.0+ CAS calculator, while also learning and exploring the mathematics necessary at VCE level in Victoria. Initially, it was expected that the calculator might be an add-on to the students' learning and that the use of the calculator might lead to a series of tricks to avoid the mathematics that we have all done in past years. However it has been found that these students fully integrated this new technology into their mathematics learning. As their teacher, I also have experienced a rapid sense of change and a new sense of what mathematics teaching is about.

The CAS-CAT Project (Computer Algebra Systems: Curricula, Assessment and Teaching Project) has emanated from the Department of Science and Mathematics Education at the University of Melbourne in conjunction with the Victorian Curriculum

* This paper has been subject to peer review.

and Assessment Authority (see HREF1). Three initial pilot schools worked with different brands of calculator, with the view to informing curriculum authorities and the wider educational community about the use of CAS in the classroom. The Project has led to a series of publications about students' attitudes and their mathematics, teacher change, and assessment issues (see Ball, Stacey & Leigh-Lancaster, 2001; Flynn, 2001; Flynn & Asp, 2002; Garner, 2002; Pierce 2001; Pierce & Stacey, 2001, 2002).

The CAS classroom

Year 11

For the first year of their VCE studies students were able to use the CAS calculator for all class work and assessments. Mathematics staff are finding that there has been a steep learning curve in the use of the calculator and, as the students progress, it is important that there is a feeling of collegiality and openness between students and teachers, and that staff feel free to visit each other's classes and learn together.

There is quite a sense of ownership in the classes that use CAS calculators, and students in the pilot study have asked for their photo to be taken for posterity.

During our time together, the CAS calculator and myself have held very much of a love/hate relationship – there are many things about the CAS which are extremely useful, yet other things which are very frustrating and misleading (Student N: Garner, 2002).

As their teacher, I have observed that this sort of response is especially noticeable in the more mathematically able students. They have a sense of what the answer should be and get very frustrated when this answer does not pop up straight away. The expectation of an answer has emerged as an important issue, coined as 'algebraic expectation', and is vital to the student's efficient use of CAS (see Pierce, 2001).

With the second year of teaching Mathematical Methods CAS Units 1 and 2, there is already a marked difference in the familiarity of using CAS in the classroom. It is easy to transfer between different aspects of symbolic, graphic and algebraic representations. The viewscreen is in constant use in all classes and the students are active participants in their own learning, having regular discussions about what CAS can and cannot do for them in their mathematics learning.

With the Year 11s I talked about instantaneous rate of change and then I looked at it on the graph. I looked at tangents. I traced the gradient. I looked at gradients in tables. I looked at gradients in the RUN menu and looked at gradients in the CAS menu. So I would have jumped to what might be perceived to be the end, to give an understanding of where we're going. It's almost like you *don't need to do* the rules as such any more. It's interesting isn't it? The rules just develop as you go along. The *understanding* of the rules and *what you need* develop amidst it. I don't feel I need to go step by step any more. And I think the kids have a better understanding of it (Teacher S: Garner, 2002).

I've found that CAS has helped a lot in my understanding of the relationship between graphs and algebra, whereas the normal graphics calculators do not help with that (Student A: Garner, 2002)

In Year 11 there appear to be two groups of students. There are those who are excellent mathematicians, and would be so in any course, who absorb the CAS into their algebraic framework. However it has been found that there are also those for whom the CAS helps them jump the hurdle of algebra that would have stopped them progressing into the study of senior mathematics. When students are initially using CAS, interesting power issues, in who 'owns' the technology, arise. I have observed that, although the use of CAS makes some students anxious, it unexpectedly empowers others.

Year 12

2002 was the first year that students in their final year of schooling sat external assessments in the new subject Mathematical Methods CAS. Sample papers and supplementary material were provided by the VCAA (see HREF 2) and students were able to revise for the exams by studying both this new material as well as the traditional sources of Mathematical Methods revision material. The students developed confidence in using their calculator and there was much discussion as to the efficient use of CAS and whether it was faster or slower in particular contexts. It was found that the decision to move from by-hand to calculator, or vice versa, differed from student to student. A point of guidance developed where the student saw a good solution as one where the CAS may be used in part (a) of a question, by hand algebra for part (b), and then picking up the CAS again for part (c). This reflects an ownership of the technology, and could be said to meet one of the goals for a CAS-active mathematics course:

To make students better users of mathematics (Stacey, Asp & McCrae, 2000, p. 250).

During an interview of teachers in the CAS-CAT Project, the following comment was recorded:

The other day I taught linear approximation in Year 12 and I just said, 'This is what you do. This is what it looks like. This is what it looks like on the calculator. This what we are looking for.' We might be looking at the small change in the y value or we might be looking at the new y value. The Year 12 kids straight away said, 'Why would you bother doing that? You can do it on the calculator.' Then I did a parametric example of it, and again they said, 'We can do it on the calculator. Why would you bother?' (Teacher S).

In an assessment task, (see HREF 1, HREF 4), students were asked to fit a mathematical model to raw data provided. It was interesting to see how these students progressed through the three sections of the task, building up to a quite symbolic section, requiring both good calculator skills as well as the insight into how and when to use those skills (see Garner, 2002). Student T, who was quite unconfident in by-hand algebraic skills, was able to attempt this third section with a confidence that surprised both him and me. It seems that the calculator gave him a tool to make sense of the algebra that he previously viewed as an unfathomable jungle. In this task one of the common solutions given by students, straight from the calculator, was in the form

of $y = e^{\left(\frac{-A^{-1}B^2 + C}{4}\right)}$. It is worthwhile considering that if we as mathematics teachers were used to a certain way of producing an answer, would we see this as a normal and acceptable form of answer? This form may not be usual for mathematicians, but is the form $y = e^{\left(\frac{-B^2 + C}{4A}\right)}$ or $y = \left(\frac{-B^2 + 4AC}{4A}\right)$ or $y = \left(\frac{B^2 - 4AC}{-4A}\right)$ any more privileged?

The students have given a personality to their calculator, and typical comments heard in class were, ‘Mr CAS made me do this’, or ‘I just love those surds’. It seems the calculator has taken on an identity all of its own and was often blamed for mistakes: ‘My calculator started doing stupid stuff in the end’ (Student T).

Assessment

School assessed coursework

Assessment tasks for the VCE (Victorian Certificate of Education) have undergone major changes since they were first introduced. The current regime of School Assessed Coursework started in 2000 (see VCAA, 2000), where internal assessment is carried out throughout the year, and then statistically moderated against the end of year exams. For the three Mathematics studies – Further Mathematics 3 & 4, Mathematical Methods 3 & 4, and Specialist Mathematics 3 & 4 – the coursework assessment tasks (SACs) comprise 40–50 minute tests, analysis tasks and an application task (see VCAA, 2001). A feature of these SACs is that technology is explicitly encouraged and included as a separate criterion in the marking scheme. Technology of any level and type is expected, meaning that computer packages, dynamic geometry systems, and computer algebra systems (CAS) are encouraged. This is as distinct from the two externally set end-of-year exams where a list of approved calculators is published annually by the Victorian Curriculum and Assessment Authority.

The difference now is that CAS technology can be used in all assessments, including externally set exams, for the approved schools with students enrolled in the subject Mathematical Methods CAS (see VCAA, 2002, pp. 7–8).

Examinations

During the exam, I found the CAS to be the perfect accessory for the multiple choice exam – the relatively short and simple nature of the questions perfectly suiting the calculator and subsequently saving me a lot of time (Student N).

It could be argued that the type of multiple choice, and other questions that can be asked with CAS, need to be different from those attempted with a graphic calculator. The viability of previous exam questions as a form of assessment using CAS has been researched (see Flynn, 2001). An analysis of 1998–2000 VCE Mathematical Methods examinations was undertaken and it was found that the questions fell into three categories: CAS No Impact Questions, CAS Impact (Useable) Questions and CAS Impact (Omit) Questions (Flynn, 2001, p. 288). In summary, there are questions that can be used in their current format, questions that can be altered for CAS use, and a

smaller proportion of questions that should be omitted from a CAS exam. This third type of question includes a test of a rule, or a solution of an equation, where the CAS takes over the algebraic skill and performs the operation in one step. An example of this is finding the derivative of the function $f(x) = e^x \sin x$, out of context. However, in reality, my students found that even though it was expected that they could solve equations or find derivatives and anti-derivatives on their calculator in one step, mistakes were often made with brackets or wrong entry of the expression, as well as misinterpretation of the output.

I think to use the CAS effectively, a sound knowledge of algebra and any processes such as differentiation and integration that will be used is needed, lest the student be confused or put off track by the dissimilar and sometimes quite complicated answers produced by the calculator's algorithms (Student A: Garner, 2002).

Even the simplest equation can be misunderstood, especially with the Casio FX 2.0 using x as the default position, so an equation in, say, m , promptly gives *no solution* if not entered correctly. To mathematics teachers this appears obvious, but to a VCE student, who is coping with many levels of information, it can confuse understanding.

Asking for the derivative of an expression like $\sqrt{x^2 - 1}$ would seem trivial with the use of a CAS that can directly give the answer. The question changed to 'Write down an expression for the derivative of $\sqrt{x^2 - 1}$ ' may still appear trivialised. However Student K entered the brackets of this expression incorrectly and discovered that her calculator gave her the answer signum (x). She panicked... Another brand of calculator would have given her a prompt that the brackets were entered incorrectly. Flynn and Asp have discussed the 'potential difficulties in setting... CAS-permitted questions for users of different CAS' (Garner, 2002).

If previous exam questions are used, with CAS merely as an add-on feature, then the assessment may not be inherently useful. However the use of CAS does not make all previous assessment items useless. The first VCE Mathematical Methods CAS exams run in 2002 (see HREF 2) show that there are a large proportion of questions that can be asked, equitably, of both CAS and non-CAS users. However it is vital that the equality of access to the questions using different brands of CAS technology is considered.

It is the vagaries in output between different brands that will require the most attention when structuring brand neutral examinations (Flynn, 2001, p. 289).

There seem to be two levels of algebraic understanding that occur with the use of CAS on an everyday basis. The first type of student uses CAS when they already could do the algebra by hand, and are using their CAS efficiently to save time, or to deal with more complicated functions. The second type of student uses the CAS to gain an algebraic understanding at a level they could not have reached before. This student will have been stopped by the barrier of algebraic rules long ago if forced to do all work by hand. It has been found in my classes that there are students who historically would never achieve in Year 12 Mathematics who have jumped the hurdle and can now deal with algebra in a real sense. This is not to say that CAS replaces knowledge and that they are just pushing buttons; rather, that the CAS has allowed entry into a world previously unattainable. It has been a pleasure to see these students gain the confidence to deal

with surds, fractions and the exchange between the representations of symbolic, numeric and graphical answers. Both types of students described above appear to be able to deal with more complicated algebraic expressions than ever before. The following quotes from three students in different phases of the Project reflect the above discussion.

It can't (usually) be used to solve problems that the user doesn't have an algebraic understanding of... the CAS may be used in the gaining of that understanding (Student C: Garner, 2002).

The CAS does help to reinforce whatever concept is being taught... It does this by doing what you tell it to do and no more... the CAS will not tidy up until the operator tells it to (Riding, 2001).

Why are you asking me does CAS make my understanding of maths better? I can DO maths now with CAS. Last year I couldn't even do it (Student M: Garner, 2002).

Student change: Explosion of methods

I got this huge disgusting answer — can I make it look nice? No it's OK — have you substituted yet? ...go on, it will cope with that disgusting expression (Teacher S, class observation, 2002).

The question often asked is whether students will lose their by-hand algebraic skills. There are two reasons for this question: one is a short-term concern and the other a wider question. The first is that while students are studying two different mathematics subjects they will need to use their CAS in Mathematical Methods, at the same time as using their graphic calculator in their other subject (in our case the more difficult subject of Specialist Mathematics). There is a concern that skills, such as factorisation or differentiation, will be hampered by lack of practice. This could be seen as a short-term issue during the phasing in of CAS use. The other issue is the broader one of what skills by-hand are privileged for their own sake, and what level of by-hand skills are needed for the student to have a feel for the algebraic form.

The VCAA has put in place assessments that are used to monitor student development of by-hand skills, as required in the outcomes for the CAS Pilot course (see HREF 2). Items used in Year 12 this year, in the *Calculus Short Review* of by-hand skills, were selected from Mathematical Methods papers CAT 2 and Exam 1, 1998–2001 (HREF 3).

Item 1

1999 – Question 15

If $f(x) = x(x^2 - 3x - 9)$ then $f'(x)$

is

- A $2x - 3$
- B $x(2x - 3)$
- C $x^2 - 3x - 9$
- D $x(2x - 3) + (x^2 - 3x - 9)$
- E $x(2x - 3) + (x^2 - 3)$

Item 2

2000 – Question 21

An anti-derivative of $\frac{1}{3x} + \sin 2x$ is

- A $\frac{-1}{3x^2} + 2 \cos 2x$
- B $\frac{1}{3} \log_e 3x - 2 \cos 2x$
- C $\frac{1}{3} \log_e x - \frac{1}{2} \cos 2x$
- D $\log_e 3x + 2 \cos 2x$
- E $\frac{-1}{3x^2} + \frac{1}{2} \cos 2x$

The first item tested the product rule, which the Casio FX 2.0 does in several steps, however this test required the completion of the items without any calculator. The Casio FX 2.0, when asked for the derivative of $x(x^2 - 3x - 9)$ gives

$$x^2 - 3x + (2x - 3)x - 9$$

expand gives

$$3x^2 - 6x - 9$$

while factor gives

$$3(x - 3)(x + 1).$$

It would seem that to recognise the correct answer D, the student would need to have a feel for the form of the product rule. It is also not unusual for students to work from likely choices and expand to work backwards to their answer. This has an impact on question design in future years.

One of the comments made in teacher discussion is that CAS students seem to be doing better at selecting the ‘candidates for the right answer’ (Teacher D) in multiple-choice questions.

The second item tested anti-differentiation, which the Casio FX 2.0 does in one step.

The results are listed in Table 1.

Table 1: Results from 26 2002 Year 12 Maths Methods CAS students without access to any calculator. (Source: Leigh-Lancaster, 2002).

Question	Examination year Question	Content	Correct answer	% of correct responses from that year's MM students	% of correct responses from 26 BGS CAS students
Item 1	1999 Question 15	Product rule	D	70	88
Item 2	2000 Question 21	Anti-differentiation	C	64	85

While not drawing huge inferences from this small, not random, sample, it could be said that the by hand skills of these students has been supported, and that CAS students have performed, on standard test items, at least as well as can be expected compared with the general Maths Methods population.

Several authors have researched the notion of algebraic insight (see Pierce, 2001, Pierce & Stacey, 2001,2002). It has been said that students will need an understanding of algebra to be able to use CAS efficiently. This impacts on the students' ability to input algebraic expressions into the CAS and to also interpret the output correctly, while dealing with the different forms of expressions that the calculator gives.

Algebraic insight is needed for doing mathematics by hand, but it is a greater part of work with CAS, partly because CAS often throws up unexpected results that would never arise by hand (Flynn & Asp, 2002,p.253).

Comments from students reflect this concern:

I have found that with long term use of CAS I have lost my ability to factorise relatively simple equations making life difficult for Specialist Maths. As a result, I think it is very necessary for students to work by hand occasionally to keep the necessary skills alive...One misplaced bracket can mean the difference between the right answer and extreme frustration (Student S) (Garner, 2002).

Without extensive by hand working in Methods during Year 11 and 12 I find myself struggling in Specialist Maths because I don't have the sound knowledge that is needed... The outcome on the CAS calculator should be expected and not used as only a resort to answering problems (Student J).

Teacher change: Ends and sides of topics

The journey of change that I found myself travelling was vastly different from what I had expected. I was surprised at how quickly the changes in my classroom occurred in the space of a year. This points to a certain level of conservatism in the original goals of the project. It was expected that in the '[r]elationship of new technology to curriculum change' that 'CAS necessitates change of methods taught (*possibly slowly*)' (author's emphasis: Stacey, Asp & McCrae, 2000, p. 245).

Rather than a slow change in methods taught, there seems to have been a roller coaster ride of fresh insights and new experiences, and a huge level of new knowledge gained,

both in calculator use and in the mathematics itself. One of the issues discussed by teachers in the initial phase of the project was whether the change in teaching that the use of CAS instigates, reflects the personality of the particular teacher.

Kendal and Stacey (1999) found that teachers will choose to highlight attributes of CAS which support their own beliefs and values about mathematics. Teachers who value routine procedures can find on a CAS a plethora of routine procedures to teach students; teachers who value insight can find many ways in which they can demonstrate links between ideas better than ever before (Stacey, Asp & McCrae, 2000, p. 248).

One comment that a group of teachers taking part in the project did make is that the success of a CAS-active curriculum 'sinks or swims with the teacher' (Teachers S, D, H, V). Teaching with CAS, appears to be a whole new way of teaching, rather than an adjunct that helps with mathematical discovery. I have described my teaching as 'teaching the ends and sides of a topic' rather than teaching procedures in an orderly fashion.

I have remembered, previously, I would have taught the rule for deriving x^n as nx^{n-1} as this big deal. I would have written it up. This is the rule. I would have done it after *First Principles*. This year I almost forgot it and I almost did it as an aside 'Yes, yes, we know that'. So somehow the rule evolves. Yes, I've stood and taught the product rule and yes, I've stood and taught the chain rule. But I've taught it in context of a problem rather than as a rule in its own right, in the context of already showing it on the calculator. So I taught the chain rule by doing it on CALC DIFF and seeing the pattern of it, the whole thing. Oh, there's the differentiation sitting there, which means differentiation of the *inner*, so that when I teach integration... In Specialist Maths where I've taught integration, by substitution methods, the CAS kids have said, 'Oh there's the derivative sitting there'. They saw it before I even commented. For the non-CAS kids, I had to go through it on the board... So the CAS kids have said 'Hey, that's easy, the derivative's just there. That's the derivative of the *inner*'. And that's not because I've said those words, it's because they'd seen that pattern (Teacher S: Garner, 2002).

I have always been confident in linking ideas, but the order of teaching topics that we have all been brought up with, seems to disappear. In the early days of teaching with CAS, as I was introducing anti-differentiation, the first thing I asked the students to do

was to enter in the CAS menu $\int x^2$. The calculator gave the response $\frac{x^3}{3}$. The students

commented straight away that this must be the opposite of differentiation. One of the students immediately gave the rule $\int x^n dx = \frac{x^{n+1}}{n+1}$. They went on to explore other

expressions confirming this rule. Discussions ensued then about the $+c$ and the dx . A comparison I would make about this method of teaching is that in teaching with graphic calculators, students can do some graphs by hand, but it is in the powerful representation of multiple graphs on the screen that students can see and understand the patterns. So it is with algebra.

Conclusions

The place of mathematics as a ‘critical filter for further education and training’ (Garner, 1997, p. 2) needs to be recognised as a context for these discussions. The use of the latest hand-held technology in the classroom can be seen to be either preparing the student for the most up to date workplace, or at the very least, providing an environment of change as education naturally evolves. In the end, this use of technology must be seen in the context of being advantageous for the students and their mathematical learning.

We must be certain, however, that students will also benefit from change: both as individuals and as future citizens of a country, which aspires to take a place in a competitive global economy, based on technology (Stacey, Asp & McCrae, 2000, p. 245).

The AAMT *Standards for Excellence* highlight these points by suggesting that excellent teachers of mathematics will:

develop informed views about relevant current trends (including... technologies)...
and use a variety of teaching practices... enhanced by available technologies
(AAMT, 2002).

Acknowledgement

The research project related to this paper is supported by the Australian Research Council SPIRT scheme, the Victorian Curriculum and Assessment Authority, the Department of Science and Mathematics Education at the University of Melbourne and the industry partners Hewlett-Packard (Australia), Shriro (Casio) and Texas Instruments.

I would also like to acknowledge and thank my colleagues in the first phase of the CAS-CAT Project: Helen Mulvogue, Val Steane and David Tynan.

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Smoke and mirrors: Integrating technology in a mathematics classroom*

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Three mathematics and one mathematics education professor were interviewed and their classroom teaching observed. Their ideas on how they perceived technology integration was investigated. I found that their perceptions of technology integration varied from professor to professor but remained consistent with their values, past experiences and needs.

Technology is starting to be seen as the driving force of progress, and education is promoted as the means of change from an industrial age to an emerging information age. Schools are under pressure to provide access to educational technology as quickly as possible (Cuban, 2001).

Despite the fact that schools are under pressure to integrate technology in their classroom and education has witnessed a multitude of both technology and innovation over the past twenty years, the impact of technological tools on education has somehow been limited (Reiser, 1987). It would be fair to say that most of the professionals of twenty years ago would not be competent and capable enough to practice their profession because of the technological revolution; but a mathematics teacher from twenty years ago would probably have no difficulty teaching today's classes. According to Kaput (1992), major limitations of technology use in classrooms are likely to be a result of limited human imagination and the constraints of old habits and social structures.

The term technology integration invokes a range of definitions, such as; 'teachers creating technology-supported projects' (Gross, Truesdale, & Bielec, 2001, p. 173), 'incorporating technology in a manner that enhances student learning' (Dockstader, 1999, p. 73), 'teachers' use of technology to accomplish curriculum or learning goals' (Driscoll, 2001). A more complex definition states that teachers are able to help students develop skills and orientations to learning with technology that are

* This paper has been subject to peer review.

appropriate for the curricular objectives and pedagogical models in place (Abrami, 2001).

Teachers' definitions of technology integration differ, and are based on the different ways that teachers use technology (Pierson, 2001). In essence, the concept or definition of technology integration is a moving target, based on predominating views of how students learn, teaching methodology, and what technologies are available to a teacher (Gross et al., 2001), but at the intersection of a teacher's content knowledge, pedagogical knowledge, and technological knowledge (Pierson, 2001). In this study, I have adopted Morton's (1996) technology integration definition, which suggests that technology integration is not simply seeing the computer as a 'tool', but rather as one of the key components of education.

In 1995, the Office of Technology Assessment identified the need to use federal leadership to integrate technology into existing national, state, and local systemic reform efforts to bring about major interrelated changes in teacher preparation, certification, and professional development, as well as curriculum and testing (Office of Technology Assessment, 1995). This marked a major shift in focus for the use of technology in the classroom, from something that we taught about to something we taught with (Reilly, 2001). This idea has been restated by different organisations. The International Society for Technology in Education (ISTE) describes technology integration in the *National Educational Technology Standards for Teachers* (NETS-T) document as follows:

Effective integration of technology is achieved when students are able to select technology tools to help them obtain information in a timely manner, analyse and synthesise the information, and present it professionally. The technology should become an integral part of how the classroom functions — as accessible as all other classroom tools (International Society for Technology in Education, 2000a, p. 6).

The National Council of Teachers of Mathematics (NCTM) also provides descriptive visions of how technology could be integrated into the teaching mathematics.

Electronic technologies — calculators and computers — are essential tools for teaching, learning, and doing mathematics. They furnish visual images of mathematical ideas, they facilitate organising and analysing data, and they compute efficiently and accurately. They can support investigation by students in every area of mathematics, including geometry, statistics, algebra, measurement, and number. When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving. (National Council of Teachers of Mathematics, 2000)

Similarly, standards for teacher competencies with technology now exist. The National Council for Accreditation of Teacher Education (NCATE) (2002) states that the new professional teacher who graduates from a professionally accredited school, college, or department of education should be able to integrate technology into instruction effectively to enhance student learning. The ISTE National Educational Technology Standards for Teachers (NETS-T) Standards focus on pre-service education, and define the fundamental concepts, knowledge, skills, and attitudes for applying technology in educational settings. The ISTE standards are purposely not designed to be examples of the integration of 'cutting edge' technologies, but instead focus on the use of

technologies that have been around for a while (International Society for Technology in Education, 2000b).

For technology integration to be successful, there needs to be a match between teacher pedagogical beliefs, the learning theory upon which the software was designed, and teacher's beliefs about the value of technology in the classroom. If not, the teacher is not as likely to integrate this software into the curriculum (Hawkins, McMillan Culp, Gilbert, Mesa & Schwarz, 1999). Nevertheless, many barriers impede teachers' efforts to integrate technology (Abrami, 2001; Hawkins et al., 1999; Sandholtz, Ringstaff & Dwyer, 1997; Schnackenberg et al., 2001). Conversely, many factors contribute to successful technology integration: leadership, vision and purpose, organised growth and experimentation, infrastructure design, professional development, community connections, software selection, and finance (Hawkins, Spielvogel & Panush, 1996).

If computer hardware, software, and infrastructure can be provided, the next most important factors which inhibit teachers' use of computers is lack of teacher training and lack of release time to learn about, practice with, or plan ways to use computers to enable teachers to develop professionally along with the technology (Schnackenberg et al., 2001; Smerdon et al., 2000). Professional development related to technology integration brings additional challenges. For example, 'a teacher attending a 3-day session on a new math program can remain fairly confident that the program will transport to any school situation and therefore, remain the same as teacher comfort level and expertise grow' (Gross et al., 2001, p. 164).

Concerns about providing teachers with appropriate preparation to use technology also permeate in-service education. Most teachers graduate from teacher preparation institutions with limited knowledge of the ways technology can be used in their professional practice. As well, most technology instruction in colleges of education involves teaching about technology as a separate subject, not teaching with technology by integrating it into other coursework to provide a model for instructional use. (Office of Technology Assessment, 1995).

It is important to understand technology-human interaction in collegiate level mathematics education (Stein, 1986; Anderson & Loftsgaarden, 1987; Tucker & Leitzel, 1995). The idea that technology is essential in teaching and learning mathematics and that it enhances students' learning is now commonly accepted. Students can learn mathematics more deeply with appropriate use of technology (Stlohl, 2001; McCoy, 1996; NCTM, 2000). Despite these research findings, there is a lack of consensus on why and how technology should be integrated into the education and what students should be taught (Wilson, 1995). According to Kaput (1992), educators need to discern what is different about the new technology and what those differences mean in terms of cognition, learning, teaching, and education, before they decide to utilise these new technological tools. The problem with reaching these goals is often identified as teachers' lack of access to, knowledge about, or skills regarding technology (Cuban, 1996; Office of Technology Assessment, 1995).

The idea that the integration of technology into undergraduate courses can provide students with knowledge and experiences that apply technology to their content areas now also has been widely accepted. Researchers showed that teachers feel they are not

prepared to effectively use technology in their classrooms (Brush, 1998; Schrum, 1999; Strudler & Wetzel, 1999). Various survey results and reports brought our attention the role of teacher education programs in preparing pre-service teachers to integrate technology in their classroom (National Council for Accreditation of Teacher Education, 1997; President's Committee of Advisors on Science and Technology, 1997). These reports and survey results informed us that there were lots of things that should be done in order to help pre-service teachers learn how to integrate technology into their classroom after graduation.

In one of my mathematics education classes, a student made a comment that inspired me to conduct this study. In class, we were talking about Richard Skemp's instrumental learning versus relational learning. She came forward and made the comment that she had been taught in an instrumental way during all her school years and she asked the class how people could expect her to teach students to develop a relational understanding. She was not sure how could she adopt her teaching philosophy and beliefs to reach this aim. This incident made me ponder about issues regarding technology integration. Do our students get enough experiences of technology integration in their classes or not? Previous research studies have showed us that students' method courses and mathematics courses affect the way in which they perceive teaching and learning mathematics (Mewborn, 1995; Strudler & Wetzel, 1999).

The use of technology in mathematics and mathematics education courses and the expectations of faculty are the main influencing factors that affect students' perception of technology integration (Brush, 1998; Schrum, 1999). One of the aims of these courses should be to help students to build confidence and skills. We know that the teachers in the school are not well trained to provide proper models for pre-service teachers (Brush, 1998). Although pre-service teachers had access to technology and necessary basic skills, they did not have many examples of actual application of technology in teaching and learning (Schrum, 1999).

In my research, I investigated three mathematics professors' and one mathematics education professor's perceptions of technology integration. These professors were selected with respect to students' evaluations and comments on their dedication to technology integration. They were interviewed and their classroom teachings were observed to identify the factors that affect their technology integration. I tried to identify the patterns of technology integration by professors, after they were first introduced to educational technology. Understanding these past patterns of adoption may give us insights into how technology integration may be utilised and/or initiated in the future.

In this study, I do not see knowledge of the technology alone as the only requirement of successful integration; their teaching skills in planning and executing mathematics lessons were also important. I believe that the use of technology itself is not sufficient evidence for a successful integration of technology on its own. I looked for evidence that shows how professors employ technology for a teaching purpose.

Albert

Albert is a Jewish man who is about fifty-five years old. He is proud to be from the sixties generation. Albert Einstein's scientific achievement was the major influencing and inspiring factor that encouraged him to be a mathematician. Teaching is very important to Albert. He has taught various courses in mathematics at different universities and has gained much respect from his teaching. Albert identifies problem posing as being of central importance in the discipline of mathematics and in the nature of mathematical thinking. His students celebrate his patience, enthusiasm, humour and dedication in the classroom. According to Albert, computers and graphing calculators are powerful educational tools that provide the possibility of amplifying students' ability by enabling them to succeed at tasks previously considered too difficult or too time-consuming. Albert sees the major role of technology in his class as that of enhancing problem posing combined with technology-assisted explorations and solutions of the problems that may lead students to think critically. He expresses his ideas by saying:

A: Now, it is easy to use technology in calculus teaching. Technology is neither cumbersome nor difficult to master; computers and graphing calculators make studying and learning mathematics interesting, interactive, and fun, while they simplify lecture preparation.

Albert's own experience with technology — observing his wife teaching online courses, and the choices of his children who specialise in computer-related jobs — were the main factors that shaped his values of technology and using technology in teaching mathematics. He explained how he integrates technology in his classroom as follows:

A: I am a technology enthusiast. I always have been and I have taught in the computer lab back to 1972. I thought differential equations and numerical analysis and that is the way that mathematics is done in real world. Engineers, economists and scientists use technology. That is the way that I found most interesting to teach. Especially nowadays, students are so weak with paper and pencil skills and at least we can choose to do something more reasonable. I have been using graphing calculators in class since 1991. That was actually to do with an experiment that I had for department using. So I used it immediately. It did have graphing capabilities, now TI-89 has also symbolic capability. It leads less drill, in fact less paper and pencil computation in computation. More stress on conceptual analysis. Students have to think on what is going on in my class and I stress on the graphs. You use pictures to understand what is going type of things.

According to Albert, the teacher is the essential ingredient in the problem posing process. He carries the following ideas to foster problem posing: (1) using problems in textbooks as a basis, and (2) using technology to promote problem posing. He believes that the primary goal of mathematics teaching should be to promote students' abstract and reasoning capabilities in the development of reasoning and solving problems. Since such reasoning contains the specific achievement of intelligence, and intelligence is the specific gift of human beings, it ought to be a centrepiece of mathematics education.

According to Albert, whether or not one believes in the inevitability of technology in classrooms, one must acknowledge that we live in a technologically-oriented society.

Albert sees the major impact of technology in teaching as motivating students to learn mathematics, and helping them to make mathematics learning a meaningful and lively experience, so that they will have a better understanding of mathematics topics.

- A: Students are always very positive about technology. It is definitely the case that technology can help people to learn mathematics. I teach the lab and I experienced that students are really turned on by computers. It is ability to make beautiful pictures, graphs and stuff, colour animations they like to move. So the computer technology is fantastic, has a huge potential to appeal students to teach mathematics. I always get positive feedback about using technology in the class and allowing them to use in the exam. They like to use graphing calculators. Especially with TI-89. They usually already have a graphing calculator like TI-83. Now, I am asking them to spend over one hundred dollars to buy TI-89. Actually one hundred and fifty dollars something like that but they still like it lot.

According to Albert, the major strength of using technology in teaching mathematics is that the teacher is no longer the centre of all knowledge within the classroom; technology enables him personally to further elaborate on what is covered during the course of one class period by delving into more detail.

- A: For example, when I am teaching the concept of derivative using the graphing calculator, I can show several equations and their tangent lines at once. This way students can see how changing the equation can change the graph and tangent line.

Albert does not like to take things for granted and he admits that human understanding is somehow limited. Albert's attitude of doing his job with little concern for philosophical questions carries over to his mathematical practice. He thinks it is important for students to make as much progress as they can on mathematical problems and especially word problems. In his class, Albert starts his class by posing questions and he tries to develop students' skills and understanding by working on calculus problems. He starts to solve questions at first heuristically, then analytically. He expresses his philosophy by saying he wears two hats: that of an engineer and that of a pure mathematician. In his problem solving, he starts to solve problems by stating that he initially puts on his engineering hat to solve the problem. He asks students, 'Why does this work?' or 'Is this conjecture always true?'. After students understand the rationale that lies behind this particular calculus problem, he moves to proof by saying, 'Let's prove this is the case always'. He finds it is important to use technology and word problems in his classes. He encourages students to use technological tools to solve calculus problems in his classes and exams. While students often view examples as models of solutions to class of problems, Albert uses them to help students understand concepts. He thinks that part of his responsibility to the students is to help them learn to read mathematical texts. He believes that the combination of software helps students to explore mathematical concepts with greater ease and discuss the results of the exploration with more instructional power. Albert does not like to lecture. He feels that it is more beneficial to the students if he answers their questions and shares perspectives with them.

Ed

Ed is a European-American man, and he is in his early sixties. He changed his major from physics to mathematics when he looked for job opportunities in physics. Later, he found that he was better suited for mathematics anyway. He believes that the availability of technology and the innovative ways that technology can be used provides an opportunity and a challenge to mathematics teachers in college classrooms to enhance the study of mathematics and enhance critical thinking. He explains his rationale for using technology as follows:

- E: With the help of technology you can change your teaching style in the sense that you can make mathematics come alive rather than just discussing abstract concepts on the blackboard, you can work some really realistic problems. It makes the material more interesting to me and you can do problems that much more interested to you.

He likes teaching because it allows him to interact with students. Ed is enthusiastic about teaching mathematics and doing mathematical research. Ed always has something interesting to tell his students in the classroom whether it is academic or not. He is also very enthusiastic about using technology in his class. He believes that student learning may be enhanced with the use of a computer and software to graph, solve equations, and do algebraic manipulation. According to Ed, experiences with both calculators and computers allow an increased separation of algorithmic processes from other aspects of knowing and doing mathematics. He uses technology in his assessments to examine the processes used by students in their mathematical investigations as well as the results. This allows him to get information to use in making his instructional decisions. He adopts the idea that using combinations of software in mathematics classrooms produces inquiring minds, thereby enhancing students' critical thinking skills.

- E: Technology doesn't do anything for you, if you don't know how to use it to increase your understanding of material you are studying and how to solve problems. It increases visual presentations you prepare. I think there is a lot more to using technology than just having one on your desktop. You have to know what to use it for, and how to use it. You have to have professors that being able to integrate it in their classrooms in a time-efficient manner that causes students get some added value in their education.

According to Ed, one of the primary goals of teaching mathematics should be to awaken each student to his or her unique gifts, questions, and contributions to the classroom discussions. We should recognise that our rapidly changing world and society have placed new challenges upon our schools as we educate and prepare students for the future. Students, if they are to be effective, should be encouraged to adjust to demands for sophisticated critical thinking, lifelong learning, and positive reaction to the changes in their environments. A teacher's responsibility includes fostering critical mathematical literacy and thus empowering students to become critical citizens in modern society. He stresses the importance of students' expectations in the creation of classroom dynamics so that every student can get benefit from it. According to him, technology in the classroom attracts different kinds of students.

- E: There are two types of students in mathematics: those who are after abstract beauty, purity, thought. They are not really concerned what it actually is used for. And those kinds of students do well in traditional classrooms where the teacher talks about all day long, etc. There are kinds of students, and I myself teach for the most part, interested in math but they are also interested in what can I do with math. And they are generally not going to be interested in taking courses in a normal way. They want to see a sort of blend between theory and practice. And it allows me to use of technology to attract them. It is a lot different, because, numerical analysis courses we offer are upper division courses.

He personally believes that the most difficult aspect of revitalising teaching mathematics is to get students to think. The students should be free from dependence on prescriptions and validations from the teacher by becoming self-reliant, autonomous learners. Instead, teachers should assist students to take the initiative to create their own solution strategies, verify their solutions, and communicate their results clearly. The teacher should encourage students to focus on concepts and to communicate mathematically in a clear manner both orally and in writing. He does not like traditional mathematics teaching that views students' understanding as a procedural and symbolic processes. He is opposed to the traditional vision that knowing a particular mathematics subject means students know a certain number of algorithmic steps that permit them to transform a symbolic expression into another symbolic expression, and the final form of that symbolic expression will be the answer to a particular problem.

According to Ed, one of the aims of education is to improve student performance using technology by centring learning around the needs of the students. He believes that our educational systems need to keep in step with the way that technology continues to change the way in which our world communicates and shares information. He desires to create a classroom where use of technology is encouraged in order to aid the learning process. Technology enables him to help students visualise and interact with mathematical concepts in new and exciting ways. According to him, graphing calculators and computers are essential tools for teaching, learning, and doing mathematics. This is particularly important for college students, because these are tools that provide organisational and visual images of abstract ideas. These representations allow students to investigate mathematics deeply by building and testing their own mathematical conjectures. He upholds the idea that students should have the opportunity to work in cooperative learning group settings throughout a mathematics course. He states his idea that working together in a mathematics class should simulate the real world business environment which requires teamwork. According to him, with the rapid advancement of technology today, it is instructor's duty to introduce students to the necessary computer knowledge and skills early in college, so that the students will be prepared to face a technologically-based society. One of his values is to have students involved in learning activities that are relevant and meaningful so that they have some direct application to their professional environments and learning goals.

He does not want students to 'just come up with an answer'. He wants them to think about how they arrive at an answer, whether it makes sense, and whether there are other possible solutions that might make sense. His technology integration in a

mathematics class involves applying ideas from various sources to create the best learning environments possible for students. He thinks about issues such as how a classroom might change or adapt when technology is integrated into the curriculum. This integration means that the curriculum and setting may also need to change to meet the opportunities that the technology may offer.

Ed wants to have sole discretion on the content of his courses by deciding the order of the topics, what the course is about, pace of the course, which textbook to use, what he wants to stress, and what technology he wants to use. While he feels he has freedom in organising his advanced level mathematics courses, he does not feel that he has the same kind of flexibility in his calculus courses: administrative decisions make an impact on the way he integrates technology into his classes.

E: In my numerical analysis courses, I have to utilise the equipment the university is going to buy you. In calculus... courses you have a syllabus, timetable, specific order of topics you have to do, and a large list that requires you to do things in a certain way, so you have a lot less flexibility to use your own professional judgment and change the class, and never quite get the best out of your students. Basically in calculus, you don't have just enough time for the way that I prefer to teach it, to do a lot of stuff.

From my point of view, I can describe his teaching style as a conductor and the curriculum as the songs which the conductor should direct. The names of the songs (mathematics topics) are predetermined; therefore, the conductor has few choices in what to conduct. But the conductor (Ed) chooses different kinds of instruments to play these songs. He has control over how these instruments perform the songs. Likewise, he is aware of the differences between students and which sort of changes in the songs will make musicians (students) pay attention to the musical tones (mathematics topics).

He is flexible and liberal in using different tones. He also gives special attention to conducting techniques, so as not to confuse the audience. Since the conductor has ultimate control over the choice of music styles, he keeps in mind that some musicians may be better at classical music than jazz, so it is his job to enhance these strengths while providing support for their weaknesses. He tries to know his students' favourite tones and attempts to tailor instruction in the same way; but, he also feels responsible for teaching them different kinds of musical tones (different approaches in problem solving, communicating with each other mathematically and improving their critical thinking).

Ed's perception of technology integration can be summarised as follows: with the help of technological tools the teacher is no longer the 'holder of all information'. Students can use technology to discover the information they need. Students are learning how to fish and are not just given the fish. Using technology can help students naturally work at their own pace and have their own needs met by taking control of their learning. He sees technology as a powerful agent to move mathematics from its current local maximum position as gatekeeper.

Patrick

Patrick is a European-American man in his late fifties. He describes his profession as one of teaching, advising, serving on doctoral committees, and helping to make departmental decisions. Though he seems to enjoy teaching, doing research is his main interest. According to Patrick, mathematics actually stands out from all other scientific knowledge with its unique characteristic, which is based on logical deductions. He believes that mathematics is the most certain of all of the sciences. Patrick seems to be bothered by the fact that, historically, there have been a lot of debates and vagueness in mathematics, which has resulted in failure to give mathematics a solid philosophical foundation. He is not happy with the fact that people would just do things in mathematics because they worked. He believes that the new developments in technology shed light on old mathematical problems and undecidable statements so that they can be solved in new ways. Therefore, we should not stop trying to give a solid philosophical foundation to mathematics.

Patrick's technology integration starts with his realisation of technology use in his family, and he became cognisant of the social implication of technology in our life. His experience with his family has helped him to realise the social dimension of technological tools and the implications of technology in education, society, and culture.

P: I realised technology as a way of multisocialisation when I observed my own family. OK. Both my wife and son get on the computer and get on the chat room and they chat with the others. My wife chats with her brothers; so it is a social media. It is replacing the telecom. Also, my son types all his papers up on the computer, he does Web searches for all his projects and he creates reports. He doesn't do a lot of math stuff on the computer because the teachers don't. She uses it as a research and presentation tool as well as the socialisation tool.

Patrick also sees mathematics as a fallible social construct, not a body of objective knowledge which does not bear social responsibility. Although he sees mathematics as a result of human creation and invention, he seems to be bothered by the fact that the existence of logical paradoxes in mathematics prevent people from viewing mathematics as a human creation. According to Patrick, the aims of teaching mathematics need to include the empowerment of students to create their own mathematical knowledge. His philosophy of teaching rests in a firm belief that students learn best when engaged in activities which are relevant to their current and future professional placement. All of his classes, whether they are at the graduate or undergraduate levels, require students to participate in collaborative projects geared towards effectively integrating technology into both teaching and learning activities. He chooses technological tools based on what he believes about learning. He feels all the important questions are really about curriculum and instruction. Patrick views technology's role in his classes as amplifying and enhancing the mathematics curriculum so that students can get maximum benefit from it. He believes that learning is not a process of transmission but a process of construction, and that students need to

have experiences where they discover information and then synthesise that knowledge with what has been previously understood.

According to Patrick, technology can be utilised in a mathematics classroom by allowing students to find multiple solutions, exploring complex ideas prior to fully understanding them, maintaining consistency between symbolic and graphical representations; and externalising visual images and thus rendering them manipulable. He is aware of the fact that there is a clear tendency among students toward uncritical acceptance of the visible graph. He is opposed to the idea that computer software moves from being one tool among many in the problem-solving process, to being the most important tool or the only tool that students can use.

P: The role of technology in my classroom is that of empowering the students. Technology is a tool that can help facilitate their learning. It helps students to take responsibility of their learning through constructing knowledge to share with others. My aim is to incorporate effective technology practices into my curriculum. I believe this will help my students achieve success regardless of the background and experiences they bring to class. I also use a variety of technologies to deliver instruction for the purpose of engaging different types of learners.

According to Patrick, the learning process is different for each student and he should be able to teach in a way that allows every student to get what he or she needs to reach their goals. He also advocates using technology to help students learn by doing. He underlines the importance of the fact that today's technology allows us to help students in their interaction with mathematical concepts in a way that was not previously possible. As an instructor, it is our responsibility to create an environment where technology can be utilised to its full potential for improving student performance. His definition of effective use of technology is to find a way to best meet the variety of needs of the students in his classrooms. He envisions explorations and applications of technology throughout his plans, thus giving the students and himself a better understanding of its capabilities. He is also aware of the danger that technology may be over-used in a classroom.

P: If the technology is the only tool in use, I believe that it is over-used. I do not use a textbook exclusively in a classroom because I do not want any one tool to be the only source of information available to my students. Still, I believe that it is a mistake to not use non-technological resources in your classroom even though students can gain access to a wide variety of resources through the use of Internet or different technological tools.

Stephen

Stephen is a European-American man in his mid-fifties. He is a professor in mathematics education. He is proud of having worked with students and teachers from kindergarten through college. He conducts research programs focussed on students' mathematics learning and the use of technology as a tool to enhance mathematical learning. According to him, the aim of the teacher education program should be to develop an educational environment wherein each student will have the opportunity to develop his/her maximum intellectual capability.

Stephen states the importance of considering educational and pedagogical implications of technological tools in his technology integration. From his perspective, the crucial step in technology integration should start with asking the following questions: How should a particular technological tool be used in the classroom and what should be taught? What should be the emphasis of instruction? He is in favour of the use of calculators in all grade levels.

- S: My terms are that calculators should be available, appropriate calculators at all science in all classes from kindergarten all the way up to doctoral studies. I think prohibiting calculators for use in math tests in college is ridiculous. It is like saying that you cannot use a microscope in biology class.

He sees teachers' adjustment of their teaching style to use technological tools more efficiently as the cornerstone for successful technology integration. Teachers need to see what new skills are required to effectively use that tool. Also teachers need to become active participants in the process to integrate technology into their teaching. Without the active, knowledgeable involvement of teachers, technology will not be integrated into the classroom routine and would merely be an addition to it.

He believes that mathematics teaching should be aimed at broadening the student's knowledge and understanding of the world, enhancing each student's ability to become an active and independent learner. He strongly holds the belief that technology has been changing the way we teach mathematics. According to him, the mathematics teacher should alter their mathematics instruction to integrate technology into mathematics instruction. In order to help students to understand mathematics concepts, the teacher should be able to use technology in their classroom demonstrations and homework assignments. The effective use of technology is very important to him.

- S: Teachers now need to look at teaching students how to make effective use of this tool, learning themselves how to make effective use of it not only for teaching but also for the management of the instruction. Let's define 'effective use' in terms of its effect. OK! What should be the effect of the use of this technology? The effect should be that students are empowered to address problems that they wouldn't be able to address without the tool, and gain a deeper understanding of the problem than they would without the tool. For some purposes, effective use could be preparation for a test and in the sense that kids still have to take tests.

According to him, the success of technology integration absolutely depends upon having a student group that understands the role of technology in mathematics education. He is concerned by the different kinds of reaction from students to his technology integration in the classes he teaches.

- S: When I use, I use calculators and computers with my elementary pre-service teachers. I don't recall getting much resistance from them in terms of calculators. The question comes out, 'Should we let kids use calculators?' With the in-service teachers, the biggest resistance to change comes from those teachers that may not have immediate access to technology in their own classrooms. So, to use the technology if they have to take that class to the lab then becomes a problem.

Conclusions and educational implications

Research findings revealed the fact that the value of personal commitment to the use of technology was a benchmark in technology integration. All professors had the knowledge of their students' abilities and needs, and necessary knowledge of technology resources in deciding how to integrate technology into their classroom. The research findings revealed the fact that the instructors do not take into account the following recent research studies' findings in their technology integration. Students, increasingly seem to trust the results from a computer or graphing calculator, even if the results contradicted other mathematical ideas they have. Many students blindly accept and copy a solution from a graphing calculator or a computer by accepting the machine's authority (Heid, 1998; Lingefjård, 2000).

The professors' Platonist, problem-solving, or instrumental views of mathematics made an impact on their technology integration (Ernest, 1988). All professors agree with the fact that mathematics is not discovered, it is a creation, and contains some brilliant notions. The mathematics education professor differs from them in his ideas; although he thinks mathematics is a human creation, he wants his students to think mathematics was discovered.

The professors all expressed that technology should be integrated when it is used in a manner to support curriculum objectives and to engage students in meaningful learning, and that technology should be part of daily classroom activities and assignments. The research findings revealed the fact that professors' understanding of technology integration is consistent with their values, past experiences and needs. Professors' personal attitudes toward technology became the benchmark in their technology integration. They expressed that students' understanding of mathematics topics were enhanced by their understanding of the topic's visual representations and that technology would best serve students' mathematical concept development. The professors expressed their concern about students blindly accepting and copying a solution from a graphing calculator or a computer by accepting the machine's authority without question. They did not address the need for recognising and addressing students' frustrations concerning the integration of technology and mathematics.

The idea that making learning goals clear by showing classroom teachers how new technologies can improve the way students learn in mathematics was adopted by all professors. The students in these professors' classes are more unlikely to utilise technology in their classroom because they have not seen a connection between their work and the technology. Although the best way to encourage pre-service teachers to embrace technology is by allowing them to have personal learning experiences on how technology can benefit themselves and their teaching, their college experiences have failed to present them with this opportunity. Professors identified the key elements in teaching with technology as not the technology itself, but how it is used and how using technology promotes larger improvements of students' understanding of mathematics. They all expressed that student-computer interaction is a complex phenomenon and the attitudes involved in this relationship are difficult to identify.

Technology integration is complex and involves the interplay of multiple factors. Although my research helped me to identify many of these factors, more research is

needed to identify which combinations of these factors need to be present to ensure successful technology integration in the mathematics classroom. I conclude there were four key factors preventing successful technology integration in a mathematics class: the students and professor did not share responsibility for learning, the professor did not consider students' voices in technology integration, technology was unevenly integrated and supported among the professors, and an inflexible, pre-determined curriculum prevented ease of technology integration.

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Looking at others and ourselves: What Australian mathematics teachers can learn from the TIMSS 1999 video study⁺

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As a tool for studying and improving teaching, video offers tremendous learning opportunities. Videotaped lessons enable teachers to efficiently and effectively ‘visit’ classrooms not only in their own school, district, or country, but also in other countries. Through observing and analysing videotaped lessons teachers can learn about *real teaching alternatives as they are actually practiced*.

As part of the Third International Mathematics and Science Study (TIMSS) 1999 Video Study⁴, over 600 mathematics lessons from seven different countries were videotaped and analysed. This paper will focus on what Australian mathematics teachers can learn from the Study, and in particular what can be learned through viewing and examining videotaped lessons from each country made publicly available. In addition, it will describe the unique software environment used for distributing these public release lessons.

Introduction

When the results of the Third International Mathematics and Science Study (TIMSS) 1999 Achievement Study were released, there was much interest in Australia about how our students fared. In particular, teachers, parents, mathematics educators, and policymakers wanted to know, ‘How did Australian students perform in mathematics compared with students internationally, and has their performance improved since the TIMSS 1995 Achievement Study?’. While the 1999 Achievement Study results provided some reassurance that our students performed relatively well, and that their scores were considerably higher in 1999 compared to 1995, those of us whose work is concerned with improving student learning wanted to know more. What does mathematics teaching look like in countries that are higher achieving than Australia?

⁺ Invited plenary address.

⁴ *Note:* The Third International Mathematics and Science Study (TIMSS) 1999 Video Study was formerly known as the Third International Mathematics and Science Study - Repeat (TIMSS-R) Video Study. At the time of writing this paper the results of the Study were not yet publicly released, therefore no mention of results can be made in this paper.

Do teachers in the higher achieving countries teach similarly or differently compared to us? And, do they teach similarly or differently compared to one another? The TIMSS 1999 Video Study was designed to seek answers to questions like these.

Background to the study

The TIMSS 1999 Video Study was a follow up and expansion of the TIMSS 1995 Video Study. More extensive and ambitious than the first study, the 1999 study investigated science teaching as well as mathematics, and sampled classrooms from more countries. Countries participating in the mathematics portion were Australia, the Czech Republic, Hong Kong SAR (Special Administrative Region of China), the Netherlands, Switzerland, and the United States. In addition, the Japanese mathematics lessons collected for the TIMSS 1995 Video Study were re-analysed using the 1999 coding scheme.

The broad purpose of the mathematics component of the 1999 Video Study was to investigate and describe teaching practices in eighth-grade mathematics in a variety of countries, including those with varying cultural traditions and with high mathematics achievement as measured by the TIMSS 1995 Achievement Study. In the 1995 Video Study involving Japan, Germany and the United States, Japan was the only high achieving country, and for some audiences it was tempting to prematurely conclude that high achievement was possible only by adopting teaching practices like those observed in Japan. The 1999 Video Study addressed this issue by sampling lessons in more countries whose students performed well on the TIMSS 1995 Achievement Study.

The mathematics component of the TIMSS 1999 Video Study included 638 eighth-grade lessons (84 from Australia). In each country, the lessons were randomly selected and videotaped across the school year. They were then coded and analysed by an international team of researchers including representatives of each country, and several specialist coding teams with different areas of mathematics expertise. Details of the results, design and methods of the TIMSS 1999 Video Study are available in the international and technical reports (Hiebert et al., forthcoming; Jacobs et al., forthcoming).

Why study mathematics teaching across cultures using video?

The videotaping, coding and analyses of hundreds of mathematics lessons from countries around the world is an elaborate and expensive undertaking. Why conduct a video survey of mathematics teaching across cultures? What can we learn from such a study?

Although direct connections between teaching and learning are difficult to draw, there is little doubt that teaching affects learning. The more we learn about teaching, the more we can learn about providing effective learning opportunities for our students. Comparing teaching across cultures has several advantages.

Comparing teaching across cultures allows us to examine our own practices more clearly. Everyday routines and practices that we engage in, in our own mathematics teaching, can become transparent and invisible (Geertz, 1984). A powerful way to

notice the practice in our own culture is through observing others. By looking inside other countries where different practices are apparent, our own teaching practices are made more visible and more open for inspection, discussion and improvement.

Comparing teaching across cultures also reveals new alternatives and stimulates discussion about teaching choices. Although some variation exists within our own culture, truly distinctive methods of teaching are the exception. Through looking at teaching in other cultures, we can increase our chances of seeing something that is really new and different. This can promote discussion about the kinds of approaches that are appropriate for achieving our learning goals. The value of the comparison lies in discovering new ideas to consider rather than simply imitating another country's methods.

Using video surveys to study teaching also has special advantages. Perhaps the greatest virtue of video is its capacity to enable a detailed examination of complex activities from different perspectives. Classrooms are complex environments and teaching is a complex activity. Video preserves classroom activity so it can be slowed down and examined multiple times, for multiple purposes, by multiple people. This makes possible detailed descriptions of many classroom lessons providing a plethora of opportunities to learn more about teaching.

An additional and important advantage of video survey is that the collection of national samples of video provides information about students' common experiences. It is not often that teaching is studied at a national level, however educational policy is most often discussed nationally. It is important to determine what teaching looks like, on average, so that national discussions of teaching can focus on what most students experience.

Examples of what we can learn

At the time of writing this paper, it was not possible to include any discussion of the results from the TIMSS 1999 Video Study due to the fact that they had not been publicly released. However, it is possible to describe some general impressions gained through my experiences as the Australian representative on the Study.

The following anecdotes are examples drawn from a collection of lessons that will be made publicly available as part of the TIMSS 1999 Video Study. This collection includes 28 lessons, four from each of the seven participating countries. These lessons will play an important role in communicating the Study results, and in providing learning opportunities for teachers around the world.

Working on a large-scale cross-cultural study of teaching provided many opportunities for learning about teaching alternatives. In selecting the following anecdotes, I chose examples of cases that I found particularly striking when I first observed them from my personal professional perspective as an Australian teacher. While it is possible to write about video observations, it is far more powerful to view them. Readers are encouraged to watch the TIMSS 1999 Video Study public release lessons in order to enhance these written descriptions. Details of how to obtain access to the lessons are provided in the final section of this paper.

Example 1:***An alternative approach for reviewing — the Czech Republic***

The international code development team, in consultation with the National Research Coordinators in each participating country, developed a code to examine the purpose of different lesson segments. Three purposes for lesson segments were defined: reviewing, introducing new content, and practicing new content. All events in a lesson were classified as one (and only one) of these three purpose types. Entire lessons might have had the same purpose throughout, or each event might have had a different purpose.

Review segments, focussed on the review or reinforcement of material presented previously. These segments typically involved the practice or application of a topic learned in a prior lesson, or the review of an idea or procedure learned previously.

In the four Australian public release lessons, review activities included warm-up problems and teacher comments reminding students of previously learned material. However, in one of the Czech Republic lessons a very different reviewing alternative can be observed.

Near the beginning of Czech Republic Lesson 1, the teacher selects a student to work on an assigned problem on the board at the front of the room. While the student works on the problem, other students are seated at their desks and solve the same problem. If they so choose, they may follow the student working at the board. The student at the board is told to give an oral explanation of the process she is using as she solves the problem. The teacher at times prompts the student for more information about the solution process by asking questions, and if the student is not able to give correct answers, other students are invited to contribute their ideas. After about six minutes, when the student has completed the problem, the teacher reviews the student's solution process, and asks the other students whether the solution is correct. The teacher then publicly allocates a grade for the student's performance, and explains the reasons for the grade given.

In her written commentary accompanying the lesson, the teacher wrote the following about this reviewing segment:

00:00:58 As a review, two students will be given an oral exam up here by the board. None of the students know which one will be called up to the board. I want them to present their knowledge by commenting, explaining to their fellow students and writing it on the board. This is an opportunity to present work in front of a group of people.

00:02:24 Marcela is supposed to apply the Pythagorean Theorem. She knows how to use it but she must also explain the reason for using it. She needs to take advantage of other geometric formulas.

The National Research Coordinator for the Czech Republic, noted these points about this segment in her commentary:

00:00:54 ...This kind of testing has two main goals. First, it's a good way to evaluate the basis of the student's grade for the performance in front of a class. And second, it gives the rest of the students an opportunity to observe how these students react to the questions. Each student at the board must answer the

questions out loud so everyone can hear them. He or she must justify their reason for the answers, and how they came up with that particular conclusion. The teacher can easily tell which part of the previous work, has been missed by the student. If the teacher gives a student some kind of hints to answer the questions, it will go against the student's final evaluation. Thanks to the comments of the students and even the teacher, many students who are behind in their study can take the opportunity to listen and observe the proper steps of solving a particular problem. Basically, those students that were either absent or simply did not understand the subject have a second chance to learn and observe the delivery of a given problem.

00:02:13 The student must explain out loud why he or she is considering certain steps. The teacher has the opportunity to find out whether the student truly understands the given topic. Sometimes a student calculates the work correctly but the teacher can ask very specific questions to reveal their knowledge level of that subject. The fact that a student must speak out loud gives them an opportunity to formulate mathematical problems with confidence, which will help them down the line to present it in a routine manner.

The comments provided by the teacher and the National Research Coordinator give a very clear rationale for this approach to reviewing. Cultural insiders from the Czech Republic who worked on the Study confirmed that this approach was common in Czech mathematics classes. Yet, to many Australian teachers this would represent a very different method of reviewing, and for some, a different set of learning objectives (including, giving students a 'second chance to learn').

Example 2:

An alternative approach for working privately — the Netherlands

Many classrooms include both whole-class discussions, in which the teacher and students interact publicly with the intent that all students participate (at least by listening), and private work, in which students complete assignments individually, or in small groups, and during which the teacher often circulates around the room and assists students who need help. A casual observation of any of the Australian or Netherlands public release lessons indicates that students in these particular lessons spent quite a lot of time working privately.

Upon closer inspection of the private work segments in the Australian and Netherlands lessons, however, it becomes apparent that teachers approach these segments in quite different ways. In the Australian examples, teachers tend to either set up a single problem for their students to investigate that requires using their existing mathematical knowledge, or they give a demonstration of some new mathematical idea or problem and assign several similar examples for students to practice.

Private work activity in the Netherlands examples looks quite different to this. For example, in the Netherlands Public Release Lesson 3, a lesson that consists almost entirely of private work, students appear to be working on a set of as many as 150 problems from a textbook. The students work at their own pace through the problems and appear to be introduced to new mathematical ideas and problem types as they progress through the textbook, rather than through any public instruction from the teacher. The teacher, during this time, is engaged in assisting individual students.

Cultural insiders from the Netherlands who worked on the Study confirmed that this approach to private work segments was common in Dutch mathematics classes. They also noted that it is not uncommon for new mathematical material to be first introduced to students in textbooks, although teachers also introduced new ideas and concepts publicly at times.

Example 3:

***An alternative approach to working on mathematical problems
— Hong Kong SAR***

An interesting observation of the 28 TIMSS 1999 Video Study mathematics public release lessons is that much of the mathematics is taught through problems. However, the ways in which teachers structure and organize their lessons around problems, the number and nature of the problems, and the ways in which the problems are worked on, varies considerably.

In Hong Kong SAR Public Release Lesson 1, the way in which the teacher worked on the problems throughout the lesson seemed quite different from similar examples I have observed in an Australian context. The lesson focussed on square numbers and square roots. The teacher provides the following rationale for her selection and coverage of these topics:

00:00:00 Students have learned about square root in elementary school. But it was focussed on practicing the calculations only. They are not very clear on many of the concepts, resulting in future difficulties when learning algebra. Therefore, I concentrated on discussing the conceptual problems with students in this lesson... Consequently, building an accurate mathematical concept.

The main focus of the lesson is on teacher-lead instruction, as well as on class discussion designed to engage students' thinking process. The reasons I focussed on these two factors: teaching in English, the tight schedule, and the time limit. Later on, I will restate the main themes in order to uphold the concepts appropriately.

The teacher's emphasis on developing students' understanding of mathematical procedures is evident in both her lesson design and delivery. As noted in her written commentary, much of the lesson involves teacher-led discussion and demonstration. Given this context, there are a number of features that make this teacher's approach particularly noteworthy. These include: a carefully chosen sequence of problems moving from simple to more challenging; frequent and deliberate questioning of students; persistent consideration and explanation of the mathematical conditions under which the procedures being developed are appropriate; and, an emphasis on precision, both in mathematical language and notation. In combination, these features appear to produce a very effective example of the direct teaching of mathematical procedures.

Learning from these and other examples

The above examples can provide numerous questions for Australian teachers to consider: How would the 'public' testing of students as a form of review be accepted in

Australian schools? Could students work independently with the support of good textbooks to learn new mathematics material? Are there particular features of the Hong Kong teacher's style that could be explored to improve the teaching of procedures in Australian schools? The feasibility of these types of approaches could make interesting discussion topics for Australian mathematics teachers.

The lessons discussed in this paper illustrate the potential to help Australian educators learn about alternative pedagogical approaches from the TIMSS 1999 Video Study, particularly through viewing and discussing the public release lessons. These lessons will provide a significant supplement to the quantitative data of the 1999 Video Study because the coding scheme could not capture many of the qualitative features of teaching.

What we can expect in the TIMSS 1999 Video Study International Report

The overriding goal of the TIMSS 1999 Video Study is to describe aspects of teaching that appear to be designed to influence students' learning opportunities. The presentation of the results in the international report of the study will be organized around three aspects of teaching that seem to both contribute to students' learning opportunities and to distinguish among countries in terms of teaching practices. These are: the way lessons are organized or structured; the nature of the content of the lessons; and the instructional practices, or ways in which the content was worked on during the lesson. In addition, details will be provided about the context of the study, including the participating teachers. The report will conclude with a discussion drawing on patterns evident across the individual features of teaching, and addressing two key questions: Are there similarities and differences in eighth-grade mathematics teaching across the seven countries? and, What are the distinctive characteristics of eighth-grade mathematics teaching in each country?

Accompanying the report will be a CD-ROM on which is presented short video clips that illustrate many of the codes used to analyse the lesson videos. In addition, the 28 public release lessons discussed earlier will be provided as a set of CD-ROMS.

An effective environment for studying video cases of mathematics teaching

The public release videos and materials are intended to augment the research findings, support teacher professional development programs, and encourage wide public discussion of teaching and how to improve it. The videos have been prepared for release in a software platform specifically designed for the study of classroom lessons, and will be available on CD-ROM format or using web technology developed by LessonLab Inc., Los Angeles (www.lessonlab.com).

LessonLab software applications integrate video, graphics, and text to create a rich learning experience. Within the software platform for each lesson there is:

- a lesson video;
- a time-linked index enabling efficient navigation around different segments of the lesson;

- time-linked transcripts in both English and the native language;
- a lesson graph displaying a plan of the lesson;
- time-linked images of textbook and worksheet pages;
- time-linked commentaries on the lesson in both English and the native language written by: the lesson teacher, the National Research Coordinator for the country, and a Researcher from the math code development team.

Other features such as discussion groups, forums, TIMSS Video Study updates, and a calendar of TIMSS Video Study events will be available through an on-line TIMSS Video Study teacher network community managed by LessonLab.

It is hoped that the TIMSS 1999 Video Study will launch a deeper and more widespread discussion among the education community about the options available for teaching mathematics, and that it might contribute to the establishment and growth of an international community dedicated to the proposition that the study and improvement of classroom teaching can and should be a collective professional enterprise.

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Properties of shape*

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Spatial or geometric thinking should be a large part of the mathematics curriculum, particularly in the early years. Shape is one aspect of Space. Some data are presented showing young children's understanding of the properties of shape and the issue of adults' understanding is also raised. Children in the trial schools in the ENRP showed markedly more learning in this area than their counterparts in the reference or 'control' schools. Concerns are raised about misunderstandings in the use of language on the part of teachers and about the use of prototypic images in presenting shape to children.

Introduction

Space, or geometry, is an important area of the curriculum that has been much neglected in the last couple of decades in our schools, right from the early years of schooling.

Geometry is grasping space... that space in which the child lives, breathes and moves. The space the child must learn to know, explore, conquer, in order to live, breathe and move better in it. (Freudenthal in National Council of Teachers of Mathematics, 1989, p. 48)

Geometric and spatial thinking are not only important in their own right but also because they provide a foundation for much mathematical learning in other areas (Clements, 2000). An example of this is the use of drawings and manipulatives in the development of understanding of fractions. The National Council of Teachers of Mathematics (NCTM, 2000) recognises its importance as a foundation.

As students become familiar with shape, structure, location, and transformations and as they develop spatial reasoning, they lay the foundation for understanding not only their spatial world but also topics in mathematics and in art, science, and social studies. (p. 97)

The authors of *A National Statement on Mathematics for Australian Schools* comment that 'we use spatial ideas for a wide variety of practical tasks' (Australian Education Council, 1990, p. 78). It is fundamental to many workplace tasks and life skills. Spatial

* This paper has been subject to peer review.

intelligence has made a large contribution to many of the top scientists and people from other fields (Dietzmann & Watters, 2000). For example, Frank Lloyd Wright had a set of geometric shapes that fascinated him as a child and were used by him throughout his life. Interestingly a book for teachers of pre-schools and the early years of school written in the 19th century (Hassell, 1888) dedicates at least a third of its content, advice and activities to the study of what we now refer to in Australia as Space. Copley (2000) notes that ‘historically, geometry was one of the first areas taught to young children’ and goes on to refer to the early kindergarten curriculum which was ‘based on the use of geometric forms and their manipulation in space’ (p. 105). One might then ask why it is that space makes up such a small part of the delivered curriculum in schools, often relegated to the end of term or the end of the year, rather than being considered an integral and important part of mathematics alongside number. However this paper is not concerned with the ‘why’; rather, having presented the argument that geometric and spatial thinking is an important foundation for future activities, this paper raises some issues of understanding in the curriculum strand of Space in the early years of schooling, and among teachers and educators.

A concern

The current focus in mathematics education is numeracy. Interestingly the Australian *Macquarie Dictionary* (1997) defines numeracy as pertaining to mathematics while the British *Oxford Dictionary* (1975) limits it to pertaining to number. The dictionary and changes within the dictionary provide an interesting perspective on education. Recently in a workshop with teachers, an activity from Burns and Tank (1988) was used to lead into a discussion of shape in the Space strand. The activity involved cutting a small kindergarten square of paper diagonally in two, then making a shape with the two triangles so formed, so that they joined along a complete edge. Figure 1 shows the types of shapes teachers made.

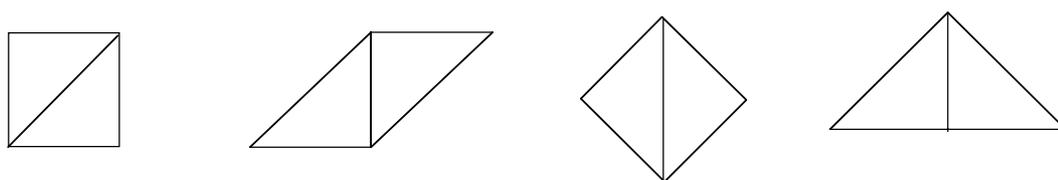


Figure 1. Shapes made from 2 pieces cut from a square.

In this, and in previous workshops where I had used this task, there was much discussion about what to call the shapes and whether there were three or four shapes made. The most interesting discussion occurred around the third shape, as the name given to it was a ‘diamond’, but there was disagreement about whether it was a diamond as it was a square. Up to this stage there had been no mention of any of the properties of the shapes in any of the discussion. The idea of properties was really forced when participants were asked to explain exactly what a diamond is. There was not agreement and, until this question, the word ‘rhombus’ had not been used. Two

days later, one of the teachers brought me the *Oxford Australian School Dictionary* (Turner & Knight, 1998) where ‘diamond’ is defined as ‘a shape with four equal sides and four angles that are not right angles’. This was not my understanding of diamond – and it did not include what to me, and all the teachers in the discussion, was a critical aspect of diamond: its orientation. Diamond is the only word used for shape where, to most people, the orientation is included. Since this definition did not sit comfortably with me (though I do not use the word diamond usually in relation to mathematical shapes) I went to my *Shorter Oxford Dictionary* and was pleased to see the definition matched my understanding.

A diamond shaped figure, *i.e.* a plane figure in the form of a section of an octahedral diamond; a rhomb (or a square) placed with its diagonals vertical and horizontal (Little, Fowler & Coulson, 1975, Volume 1, p. 540).

Dictionaries change to reflect changes in the use of language in the community, but I wonder where the meaning of ‘diamond’ came from in the *Australian School Dictionary*. The *Macquarie Dictionary* (1997) mirrors the *Oxford* definition. Booker, Bond, Briggs and Davey (1997), in a book written for teachers and trainee teachers, claim that a diamond has four equal sides and no right angles but do not discuss orientation, although every diamond they have drawn has the ‘correct’ orientation. This has also been reflected in some primary mathematical textbooks (e.g. Lilburn & Rawson, 2000).

Children’s understanding of shape

This experience raises some issues related to teaching in this area of the curriculum, and also raises the question of what understandings children have about shape generally. Geometric and spatial thinking of course goes well beyond shape, though shape will be the focus of this paper. Clements (2000) claims that teachers and curriculum writers often assume that children in early childhood classes arrive with little or no knowledge about geometric figures and thus school adds little to their knowledge, with teachers rather affirming what they often already know. He continues by describing some of the knowledge young children have shown in research projects: over 90% of 3–6 year-olds could identify circles quite accurately, differentiating them from ellipses as well as other shapes, over 80% of 4–6 year-olds could identify squares fairly well, though for triangles the figure dropped to about 60% correct. For all these identifications a range of shapes, including non-prototypic shapes, was used. While the children did progress steadily at school the growth was not remarkable and the differences between the school children and the pre-school children were not great.

In the Early Numeracy Research Project (ENRP), a major three-year Victorian initiative (Clarke, 2001), two of the nine domains studied were from the Space curriculum area. One of these was Properties of shape. The study of properties of shape typically involves both two dimensional and three dimensional shapes. For example, for Level 2 for Shape and Space (end of Year 2) (Board of Studies, 2000) the stated curriculum foci read

At this level students develop the ability to identify, describe and compare two-dimensional shapes and three-dimensional objects with reference to their component parts.

They are increasingly able to recognise, represent and make simple shapes and objects from visual and oral descriptions. They investigate and produce patterns using simple repetition and movement of shapes (p. 40).

The domain of properties of shape studied in the ENRP, on which data were collected on children's understanding, was restricted to two-dimensional shapes.

One of the best known hierarchies for understanding aspects of geometric thinking is that of van Hiele (e.g. 1986). The transition between the first two van Hiele levels, Visual, and Descriptive/Analytic has been described as

children first find salient the overall appearance of shapes... and then come to recognise shapes as carriers of properties such as the number of sides of a figure or the measure of its angles (Lehrer, Jenkins & Osana, 1998, p. 138).

In the junior primary grades, most emphasis is on these first two van Hiele levels and usually it is not until later that there is a focus on the third level, Abstract/Relational, which involves abstract definitions, a hierarchy, and reasoning about the properties of classes. Van Hiele's view of these levels was linear, suggesting each level develops fully and engenders the beginning of the next level. Clements (2000) has hypothesised a more synergistic view of the levels, with all levels growing though at various stages one may be more dominant, but with the earlier levels not disappearing as new ideas gain ascendancy. This has resonance with those of us who are comfortable operating at higher levels but often know a shape because it looks like an image we have, and thus we just recognise it rather than because we have analysed its properties.

Properties of shape domain within the ENRP

Within each of the domains studied in the ENRP, four to six growth points, sometimes described as key stepping stones in learning, were used to indicate a learning trajectory. These growth points went beyond what is described in the curriculum for the first three years of school as the intention was to not limit learning by setting a low ceiling. In considering the Growth Points for Properties of Shape (see Figure 2), it is clear that only Growth Point 4 is starting to relate to van Hiele level 3. Growth Point 1 relates to the visual while Growth Points 2 and 3 focus on developing properties of shape.

There are many questions that can be asked of children and many tasks that could be given to observe children and infer their understanding of properties of shape. The assessment in the ENRP was a one-to-one interview assessment, which included handling materials. The tasks used in the properties of shape section of the interview firstly require students to sort a variety of shapes (several triangles, a rectangle, a square, a circle) into groups. The students are then asked to name the various shapes and to describe the ways in which the groups vary. Finally they are presented with a range of triangles and non-triangles and asked to distinguish between them, justifying their decisions (if possible) in terms of properties.

- | |
|---|
| <p>0. Not apparent
<i>Not yet able to recognise and match simple shapes.</i></p> <p>1. Holistic recognition of shape
<i>Can recognise resemblances and match some simple shapes, using standard "prototypes".</i></p> <p>2. Classification of shapes, attending to visual features
<i>Can sort and compare shapes, using some geometrical language to describe features.</i></p> <p>3. Identification of "classes of shapes" by some properties
<i>Uses properties of shapes to classify shapes into classes, using appropriate language.</i></p> <p>4. Definition of shapes using properties
<i>States and understands conditions for defining key shapes.</i></p> |
|---|

Figure 2. ENRP Growth Points for the domain of Properties of Shape.

Some Prep (Grade 0) children arrive at school with considerable knowledge of shapes and their properties. Over one quarter were able to name correctly squares, circles, rectangles and a range of triangles (including naming the non-prototypical right-angled triangle as a triangle), with over 60% able to do so by the end of the year. This seems less than the proportion of children who could identify circles, squares and triangles from drawings of multiple shapes with both examples and non-examples (Clements, 2000), but the ENRP assessment did include rectangles as well.

Over the three years of the ENRP (1999–2001) the teachers in the trial schools participated in an extensive professional development program and used a variety of ongoing assessment tasks in their classes as well as collecting assessment data by using the ENRP interview. In the matched reference schools only the interview assessment was used.

In Figure 3 the spread of children across the growth points at different stages and grades is shown. A vertical line at any of the four grade and time points shows how the children spread across the growth points. The Space domain was only assessed in 2000 and 2001. The children arriving at school in 2000 in Grade Prep are shown at the start and end of the year and the children in Grade 1 at the end of 2000 are shown then progressing to the end of Grade 2 in 2001. After two years in the ENRP over 68% of the Grade 2 children and 48% of Grade 1 children at the end of the year were using properties to classify shapes into classes and, at least with these two dimensional shapes, showing some characteristics of van Hiele level 2. In the reference schools, where teachers were not involved in the ENRP professional development and therefore are typical of Victorian schools generally, only 32% of children had achieved this by the end of Grade 2. It is of interest though that there was no significant gender differentiation in the spread of responses either on school entry where, if anything, girls were very slightly ahead of the boys, or as a result of teaching in either the trial or reference schools.

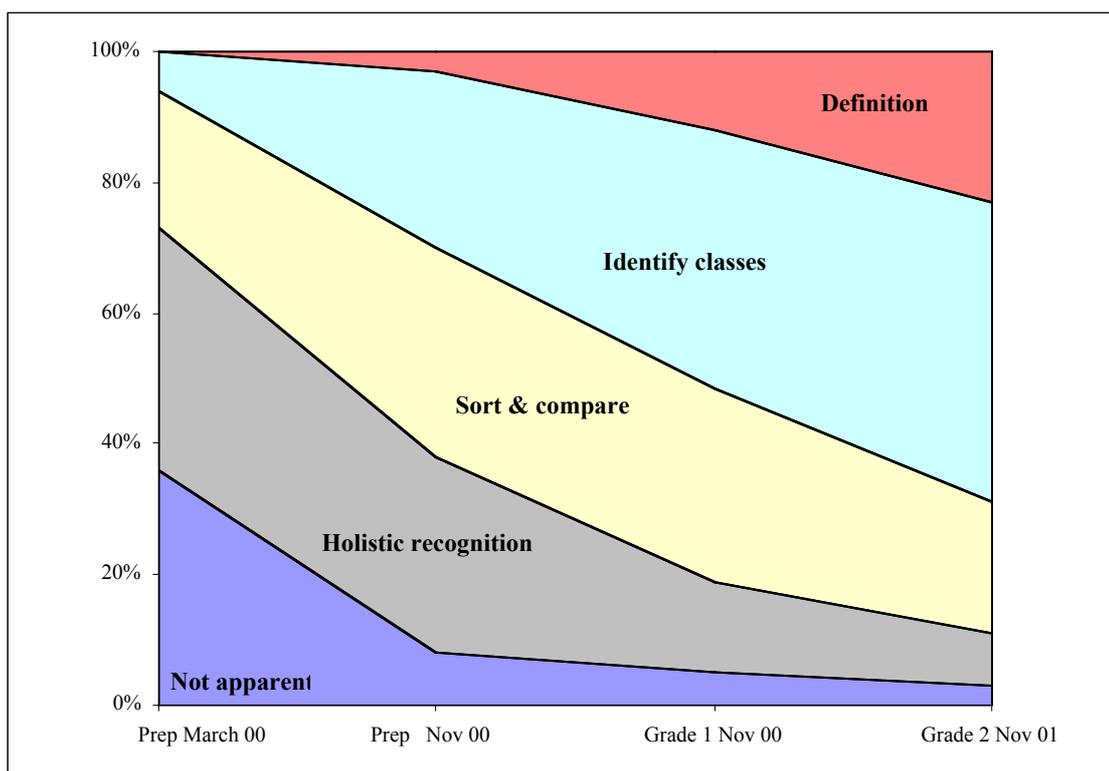


Figure 3. Trial school students (%) achieving Properties of Shape growth points over time.

The children in the interview task were asked to sort some shapes, then explain their groups. It is of interest that a number of children classified the groups into squares, rectangles, circles, triangles and other, with the other shapes being right-angled triangles with the base half of the height. The explanations included things like, 'They are like triangles but they are too long and pointy' and, indicating the width, 'There is not enough here'. Some children even named them 'half triangles' since two fitted together to make a 'triangle'. The prototypic view of a triangle as an equilateral triangle or an isosceles right-angled triangle was often evident. In a later part of the interview concerned with visualisation and orientation, the prototypic view that a triangle has a horizontal base also showed. As one child commented, pointing to an equilateral triangle point down, 'Well if you turned this one around it would be a triangle'. While textbooks at the early years of school do show the triangle point down they also tend more often to have very prototypic presentations of triangles (e.g. Parker, McSweeney & Sheehan, 1997; Williams, 1999).

All of the children differentiated between squares and rectangles. At this early stage of geometric understanding the hierarchy of shapes is not realised and children see rectangles as non-squares even though a square is a special rectangle. Many children's books make this distinction, also treating rectangle and square as disjoint shapes. Some textbooks for primary school children have tried to overcome the classification of squares being a special type of rectangle by using oblong as a shape that is a rectangle but not a square (e.g. Lilburn & Rawson, 2000) although in the same book another page asks the teacher to 'cut cards into square, rectangular or circular shapes' (p. 13). In

the ENRP interview, testing children usually used the word ‘rectangle’ and rarely ‘oblong’. Including ‘oblong’ as a term in the mathematics classroom may add more confusion, but it is a word that is also used sometimes outside the mathematics classroom. The use of the word ‘diamond’, however, is more likely to add confusion and would be better not used in mathematics classes. While the definition is clear in some dictionaries, agreeing with common understanding that links orientation to the shape, it makes it the only shape where orientation matters, and not all books use this definition. Children are capable instead of using the word ‘rhombus’.

Adults’ understanding

Because Space/Geometry has been such a reduced aspect of the curriculum for many years, often teachers do not themselves have the mathematical understanding of properties of shape. In the workshops previously mentioned, many teachers have expressed surprise and sometimes disagreement when it has been suggested that a square is also a rectangle. The prototypic view of shape rather than an understanding of properties is also common. When asked in a workshop to draw a hexagon, it is rare to find a teacher who does not draw something like a regular hexagon. Indeed, university students with at least mathematics minors, when given this task, tend to draw regular hexagons as in the left of Figure 4, usually with the same orientation. One student was even seen to erase one part of it because it was not close enough to the regular hexagon. Again there has been surprise and even disbelief when it is suggested to them that the shape on the right is also a hexagon.

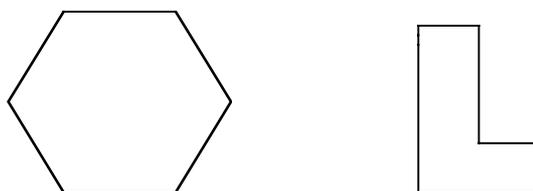


Figure 4. Hexagons.

In reflection on my own understanding I can see aspects of many of van Hiele’s levels in my thinking on different occasions. It is clear also that the hierarchy of shapes and their properties is not always well understood by those responsible for children’s education.

Implications

Improving mathematics teaching requires improving knowledge of mathematics itself, knowledge of children’ learning of mathematics, knowledge of the particular children in the class and knowledge of mathematics pedagogy (Horne, Cheeseman, Clarke, Gronn & McDonough, 2002). Shapes should be presented to children in a range of forms with particular emphasis on extending the boundaries of understanding by avoiding the

prototypic presentations. Textbook writers have often reinforced the prototypes and need to revise their approach.

With regard to language and classification, young children do not appear to have difficulty with the names of dinosaurs or with the idea that a ‘stegosaur’ is both that and also a dinosaur. Why then do we shy away from using correct mathematical terms and making children aware that one shape can have more than one name? A square is a rectangle but can also be called rhombus, parallelogram, and quadrilateral. To develop these ideas children need a lot of experiences with both examples of the concept and non-examples of the concept, presented in a variety of ways. It is unnecessary to add the confusion of naming a shape defined by its orientation, such as diamond, or a shape such as rhomboid that can so easily be confused with rhombus. Words like parallelogram, rhombus, quadrilateral, pentagon and hexagon should be used as a natural part of the mathematics classroom. Children’s understanding of the concepts will grow as they meet the shapes in connection with the language through a variety of experiences. Why not use many words to describe rather than trying to make disjointed classifications?

The ENRP has shown in comparison of trial and reference school data that attention to geometry in the curriculum can make a difference in children’s learning and they are capable of developing in geometry beyond the expectations of the last few decades. Space is an important area of numeracy and needs attention. The mathematical focus of activities used in classes needs to be clear rather than space just providing some fun activities for Friday afternoon or the end of term.

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The Victorian Curriculum and Assessment Authority (VCAA) Mathematical Methods (CAS) pilot study 2001–2005*

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The use of computer algebra systems (CAS) in secondary school mathematics has now become much more accessible. The focus of earlier work with CAS has generally been with respect to pedagogical and curriculum issues. As familiarity with CAS has developed in senior secondary mathematics contexts around the world, education systems and organisations have responded in various ways to the increasing availability of CAS and its impact on assessment. Central to consideration of such issues are values and beliefs about the nature of mathematics and mathematical activity, the notion of congruence between pedagogy, curriculum and assessment, the availability of suitable resources, and teacher professional development. This paper briefly describes the Victorian Curriculum and Assessment Authority (VCAA) Mathematical Methods (CAS) pilot study, reports on developments to date, and outlines materials from the pilot that are available from the VCAA website.

Background

Interest in, and the use of technology such as graphics calculators and computer algebra systems (CAS) in senior secondary mathematics has increased significantly around the world over the last decade. The affordability and portability of calculators compared with computer-based software such as CAS has led, in most cases, to the former being taken up more widely as a tool to support and enhance the teaching and learning of mathematics. Over the past few years there has been a convergence of hand-held technology and CAS, there are now several different brands and models of hand-held and computer-based CAS available throughout the world. This technology is becoming increasingly affordable, with some hand-held CAS calculators such as the HP-40G available at comparable cost to the corresponding graphics calculator and

* This paper has been subject to peer review.

some computer-based CAS software being available for less than the cost of graphics calculators. CAS have been widely used for some time in senior secondary mathematics classes in countries such as Austria, Denmark, France, the Netherlands and Germany and increasingly in other countries such as Australia, Israel and the United States.

Their use is permitted in the *Baccalauréat Général Mathématiques* examination in France, and parts of the College Board *Advanced Placement Calculus* examination in the United States. The *Danish Baccalaureate Mathematics* can be taken with a non-CAS or CAS version of its examination. This includes a collection of common questions and alternative versions of some questions for students who undertake a CAS based implementation of the Danish *Baccalaureate Mathematics*. The diploma review committee for the International Baccalaureate (IBO) has approved a pilot course for mathematics that requires the use of computer algebra systems. The pilot course will be developed in parallel with the mathematics Higher Level course as part of the current IB review process.

While earlier considerations on the use of CAS in mathematics education focussed on pedagogical and curriculum issues, these issues do not arise in isolation from assessment (see Leigh-Lancaster & Stephens, 1997, 2001). The notion of *congruence* between pedagogy, curriculum and assessment is a central part of the discourse on the use of such technology (see Leigh-Lancaster, 2000, HREF1).

The use of technology in the senior mathematics curriculum, and end of secondary schooling mathematics examinations in Victoria, has evolved over the last several decades as different technologies have become more widely available and integrated into mainstream teaching and learning practice:

- 1970 – slide rule and four figure mathematical tables;
- 1978 – scientific calculators;
- 1997/8 – approved graphics calculators permitted (examinations graphics calculator ‘neutral’);
- 1999 – ‘assumed access’ for graphics calculators in Mathematical Methods and Specialist Mathematics examinations, permitted for Further Mathematics examinations;
- 2000 – ‘assumed access’ for graphics calculators in all mathematics examinations, examinations for revised Victorian Certificate of Education (VCE) Mathematics study 2000–5 incorporating some graphics calculator ‘active’ questions;
- January 2001 – Mathematical Methods (CAS) pilot study, ‘assumed access’ for approved CAS in pilot examinations.

The current pilot study takes place within a broader VCAA curriculum policy context for the development and use of technology in senior secondary education, including formal assessment.

The VCAA Mathematical Methods (CAS) pilot study 2001–2005

Mathematical Methods and Mathematical Methods (CAS) Units 1–4 correspond with Victorian senior second mathematics courses which cover material related to functions, algebra, calculus and probability and are used to meet entry requirements for university study in disciplines such as science, medicine and economics. The former assumes student access to an approved graphics calculator, while the latter assumes access to an approved CAS. Units 1 and 2 are typically undertaken in Year 11, while Units 3 and 4 are typically undertaken in Year 12. For Units 3 and 4, coursework assessment is worth 34% of the final result, and two equally weighted one and a half hour end of year examinations: Examination 1 *Facts, skills and applications* – multiple choice and short answers questions, and Examination 2 *Analysis task* – multi-stage extended response questions of increasing complexity, are together worth 66% of the final result. Coursework assessment is statistically moderated with respect to these examinations. The corresponding examinations for both Mathematical Methods and Mathematical Methods (CAS) courses have the same structure and are held concurrently. There is substantial common material between corresponding examinations for each course, as well as distinctive material appropriate to each course.

Mathematical Methods (CAS) Units 1–4 is an accredited pilot study of the Victorian Curriculum and Assessment Authority for the period from January 2001 – December 2005. The pilot study is monitored and evaluated as part of the ongoing review and accreditation of VCE studies, and, following on from a successful conclusion to the pilot, there would likely be a subsequent period of overlapping accreditation for the revised Mathematical Methods and Mathematical Methods (CAS) courses. Details of the pilot, including the study design for Units 1–4, sample examinations and other teacher resources, can be accessed from the VCAA website ([HREF2](#)).

The first phase of the pilot study 2000–2002, involved students from three Stage 1 schools, and was implemented in conjunction with the CAS-CAT project, a research partnership between the VCAA, the Department of Science and Mathematics Education at the University of Melbourne, and three calculator companies (CASIO, Hewlett-Packard and Texas Instruments). The CAS-CAT project has been funded by a major Commonwealth Australian Research Council (ARC) Strategic Partnership with Industry Research and Training (SPIRT) grant ([HREF3](#)). In November 2002, about 80 students from the three Stage 1 schools sat final Mathematical Methods (CAS) Unit 3 and 4 examinations, for which student access to an approved CAS calculator was assumed.

The second stage, the expanded pilot study 2001–2005, incorporates the original three schools (implementing Mathematical Methods (CAS) Units 1 and 2 from 2001 and Units 3 and 4 from 2002) and includes two additional groups: nine Stage 2 schools implementing Units 1 and 2 from 2002 and Units 3 and 4 from 2003, and a further seven Stage 3 schools implementing Units 1 and 2 from 2002 and Units 3 and 4 from 2004 ([HREF1](#)). The schools in the expanded pilot include metropolitan and regional schools from government, catholic and independent sectors, using a range of different CAS. Thus, there will be several hundred students enrolled in the expanded pilot from

2003, and there may be the opportunity for some further expansion of the pilot from 2004.

The Danish experience, and the initial experience of the expanded pilot, strongly indicates that the rate of uptake of CAS based courses during a period of overlapping accreditation of CAS and non-CAS based courses would be gradual, while teachers developed their familiarity with the CAS based course, and their own confidence with the use of CAS and further course specific resources are developed.

Developments

Early use of CAS by teachers and students in Victorian senior secondary mathematics focussed on its use as a pedagogical tool for improving student learning within existing courses, and to support student work in responding to the complexity and generality of mathematics in extended investigations modelling and problem-solving tasks such as the centrally set, but school assessed, extended VCE mathematics common assessment tasks (see, for example, Tynan, 1991; Woods, 1994; Delbosc and Leigh-Lancaster, 1995).

The revised VCE mathematics study, implemented from 2000, is explicit about the effective and appropriate use of technology to produce results which support learning mathematics and its application in different contexts:

The appropriate use of technology to support and develop the teaching and learning of mathematics is to be incorporated throughout each unit and course. This will include the use of some of the following technologies for various areas of study or topics: graphics calculators, spreadsheets, graphing packages, dynamic geometry systems, statistical analysis systems, and computer algebra systems. In particular, students are encouraged to use graphics calculators, spreadsheets or statistical software for probability and statistics related areas of study, and graphics calculators, dynamic geometry systems, graphing packages or computer algebra systems in the remaining areas of study systems both in the learning of new material and the application of this material in a variety of different contexts. (Board of Studies, p. 12, 1999).

The Mathematical Methods (CAS) pilot develops these considerations with respect to congruence between pedagogy, curriculum and assessment for computer algebra system technology. Consultation with universities and the Victorian Tertiary Admissions Centre (VTAC) took place throughout the development and accreditation of Mathematical Methods (CAS) Units 1–4 for the pilot study and in March 2001, VTAC informed the VCAA that the pilot study design had been approved by all universities for prerequisite purposes from 2003.

In May 2002, the Victorian Qualifications Authority (VQA) endorsed a VCAA recommendation to extend the accreditation period of the current VCE mathematics study to 31 December 2005, following the initial work of the VCAA Mathematics Expert Studies Committee, including representatives from universities, in 2002. This decision aligns the accreditation period of the current study with that of the Mathematical Methods (CAS) pilot and enables the review of Mathematical Methods and Mathematical Methods (CAS) to take place concurrently, informed by data from several

years of pilot program implementation, as well as further international data and experience.

Various Australian mathematics educators and researchers have published CAS related articles over the past few years (see, for example, Kissane, Bradley & Kemp, 1997; Leigh-Lancaster, 1998; Leigh-Lancaster & Stephens, 1999; Stacey, McCrae, Chick, Asp & Leigh-Lancaster, 2000; Ball, Leigh-Lancaster : Stacey, 2001). In 2002, teachers from the expanded pilot have written papers for the 2003 Australian Association of Mathematics Teachers (AAMT) *Mathematics ~ making waves* biennial conference (see Garner 2002b; Tynan 2002) and also for the Mathematical Association of Victoria annual conference in December 2002 (see Garner, 2002a; McNamara & Shardlow, 2002; Robertson & Karanikolas, 2002; Coffey & Barber, 2002; Moya, 2002). These papers provide a rich insight into the practical experience of teachers and students using CAS in the teaching, learning and assessment of a course designed with assumed student access to CAS technology — the Mathematical Methods (CAS) pilot.

During 2001 and 2002, two teachers from each of the nineteen expanded pilot schools attended five comprehensive full day VCAA professional development workshops covering the areas of study and assessment for the course across Units 1–4. Related discussion papers, learning activities and assessment materials from these workshops are available from the VCAA website. Several specific CAS based networks have been set up by the teachers involved and, from 2003, the sharing of resources, ideas and approaches by teachers as they further develop and refine their course implementations will likely be a focus for professional development activity. An important reflection by teachers is that the use of CAS in the professional development was a powerful tool for refining and enhancing their mathematical knowledge and understanding, and that the opportunity it afforded them to focus on this is highly valued. Teachers also reported that student use of CAS supported engagement with more complex material, more independent learning approach by students, a more in depth treatment of the concepts of variable a function, and enabled a greater degree of generality. The VCAA has gathered data from the schools in the expanded pilot through responses to a pilot study implementation report where teachers have provided feedback on:

- areas of study and content;
- the outcomes and related key knowledge and key skills;
- assessment of student learning (including the development of by-hand algebra and calculus skills);
- course management (teaching sequence and implementation);
- recommendations for material to be prepared as advice for teachers;
- areas for needed professional development and resource development;
- CAS-specific issues (with respect to the particular CAS used); and
- other issues, as identified by the teachers.

This data, as well as data from the research project, VCAA examinations and coursework assessment, is an important part of monitoring the implementation of the

pilot study. It will also inform the preparation of further advice and materials of the study and the review process.

Resources

The main approach to resource development has been to supplement existing texts and VCAA resources for the Mathematical Methods course with material that is particularly appropriate for CAS-based work. The Mathematical Methods (CAS) pilot section of the VCAA website (HREF2) contains a substantial range of resources to support teachers in their implementation of the pilot study. These incorporate the materials used for the professional development workshops in 2001 and 2002 and include:

- sample examination papers, supplementary questions with solutions, comments and advice (multiple choice, short answer and extended response items);
- discussion papers and teaching approaches (for example, implementation advice to teachers, CAS functionality, functional equations, the modulus function and derivatives of combined functions);
- sample learning activities and tasks (for example, tests, assignment, modelling activities, application tasks);
- reference materials (publications, lists of relevant Australian conference articles, websites);
- links to the CAS-CAT project website (project resources, discussion papers, publications).

The VCAA website will continue to be progressively updated to include the 2002 examinations and corresponding report to teachers as well as further advice and teacher support materials as they are developed.

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'Concentrating the faculties': Mental arithmetic in pre-1970s classrooms in Queensland*

Geoffrey R. Morgan

Mental computation is receiving renewed emphasis in mathematics classrooms. Students should view calculating mentally as the approach to be attempted first, before rushing to paper-and-pencil or a calculator. Many of the current beliefs about mental computation have their echoes in the past. This is despite major differences in the way mental computation is now envisaged compared to that for mental arithmetic prior to the late 1960s, a time when mental calculation was last given explicit reference in Queensland syllabuses. This paper presents an insight into past beliefs about mental arithmetic, and provides reasons for the non-ascendency of those that are in harmony with the present.

Introduction

Although the mathematics syllabuses in Queensland from 1966 have not explicitly emphasised the mental calculation of exact answers, mathematics educators, from the mid-1980s, have been taking a renewed interest in calculating mentally. Although not neglecting the correctness of the answer, it is now recommended that the emphasis be placed on the mental processes employed. It is this which distinguishes mental computation from earlier considerations in which the correctness of the answer was of prime concern – that is, to distinguish *mental computation* from *mental arithmetic*.

The resurgence of interest in mental calculation has its origins in a number of sources. Calculating mentally remains a viable alternative, despite the availability of various electronic calculating devices. It continues to be the major form of calculation used in every-day life. Coupled with this recognition is the realisation that the standard written algorithms – a major focus of primary school mathematics – are seldom used outside the classroom. Further, there is sufficient research evidence now available to suggest that an overemphasis on written methods may reduce an individual's development of flexible mental strategies. Hence it is argued that paper-and-pencil skills should receive

* This paper has been subject to peer review.

decreased attention in schools. Mental computation is no longer viewed as an end in itself, but rather as a means for promoting an individual's ability to think mathematically, the novel facet of the current resurgence of interest (Reys & Barger, 1994).

Although teacher interest in mental computation is gradually increasing, anecdotal evidence suggests that little attention continues to be given to actually *teaching* mental computation in Queensland primary classrooms. It is only in some of the sourcebooks that have been written for the current mathematics syllabus, particularly in that for Year 5, that specific references to mental computation have been included. However, the philosophy of the *Years 1 to 10 Mathematics Syllabus* (Department of Education, 1987) is not in conflict with the recommendations of mathematics educators concerning how mental computation should be taught. Nevertheless, the significant debate necessary for curriculum change in Queensland, and in Australian schools in general, has not yet occurred to any degree. This, despite the imminent implementation of a revised syllabus that brings mental computation to the forefront of computational processes.

Such a debate can be informed by an understanding of the past — in this instance, the nature of mental arithmetic in Queensland primary classrooms prior to the 1966–68 New Mathematics syllabuses. The analysis that follows is intended to provide a summary of key similarities and differences between the beliefs and teaching practices related to mental computation currently advocated by mathematics educators, and those of Queensland teachers across the period 1860–1965. Consideration has been given to: (a) the emphasis placed on the mental calculation of exact answers, (b) the roles ascribed to such calculation, (c) the nature of these calculations, and (d) the approaches to teaching mental arithmetic.

The emphasis placed on mental arithmetic

Should not all arithmetic be mental, and the pencil called into requisition only when the numbers are large? Are not slate and paper used merely to lessen the strain on the memory? (Mr District Inspector Mutch, 1906, p. 63)

It is apparent that the importance of mental calculation has long been recognised by at least some Queensland educators. Prominence was given to the calculation of exact answers mentally in each of the Queensland syllabuses from 1860 to 1964, albeit under various headings, which included *mental exercises*, *mental*, *mental work*, *oral work*, *mental and oral work*, *oral arithmetic*, and *mental arithmetic*. From that of 1904, each syllabus emphasised that 'mental calculations should be the basis of all instruction' ('Schedule XIV,' 1904, p. 201). The 1914 and 1930 Syllabuses suggested that 'new types of problems should invariably be introduced in this way' (Department of Public Instruction, 1914, p. 61; 1930, p. 31), whereas the 1952 Syllabus stressed that 'all written work should be preceded by introductory oral [that is, mental] exercises' (Department of Public Instruction, 1952, p. 2).

Despite the prominence given to mental arithmetic in the syllabus documents and the exhortations of District Inspectors of Schools for teachers to implement their intent, the generally low standard of mental calculation was most commonly attributed to its

not receiving sufficient regular and systematic treatment during the period 1860–1965. Mental arithmetic appeared to be the *bête noir* of many teachers ('Mental Arithmetic,' 1927), an outcome of the characteristics ascribed to it and of the ways in which it was taught. Nonetheless, it can be concluded that mental arithmetic received some emphasis by teachers during the period 1860–1965.

Roles of mental arithmetic

The various roles attributed to mental arithmetic from 1860–1965 in Queensland are associated with its social and pedagogical usefulness, and with its perceived usefulness for strengthening the mind. However, specific recognition of its social usefulness was given only in the 1952 and 1964 Syllabuses, with the former acknowledging that 'oral arithmetic is more commonly used in after-school life than written arithmetic' (Department of Education, 1952, p. 2). Nonetheless, the social value of mental arithmetic was recognised by the authors of textbooks used in Queensland schools from 1860. Park (1879), for example, stressed that mental arithmetic was a 'subject of great practical importance' (p. 42), a conviction often reiterated by District Inspectors in their annual reports. Representative of these was that of Bevington (1926) who commented that 'in domestic and mercantile transactions, in calculating about farms, &c., mental exercises are so frequent that it seems to be absolutely essential that children [should] be trained to calculate quickly and accurately' (p. 80).

However, it is in relation to the pedagogical usefulness of calculating mentally that distinctions can more clearly be drawn between the beliefs presently held by mathematics educators and those of the past. In contrast to past beliefs and practices, which focussed on gaining answers speedily and accurately, it is now recommended that the focus should be on the mental processes involved. Such a focus would allow mental computation to be used as a tool to facilitate the meaningful development of mathematical concepts and skills.

While occasional recognition was given to mental arithmetic as a means for 'making scholars think clearly and systematically about number' ('Teaching Hints: Arithmetic,' 1908, p. 15), the primary pedagogical function of mental work during the period 1860–1965 was to familiarise students with the arithmetical operations prior to an emphasis on paper-and-pencil calculation. This mental-written sequence was encapsulated in the 1904 Syllabus by its stressing that 'the pupils should be made familiar by mental exercises with the principles underlying every process before the written work is undertaken' ('Schedule XIV,' 1904, p. 201), a sequence that was embodied in the spiral nature of the 1930 Syllabus. However, the failure on the part of teachers to sufficiently model their mental arithmetic examples on the written work that was to follow was a regular criticism of District Inspectors in their annual reports (Morgan, 1999).

Mental work was also considered to be an effective means for cultivating speed and accuracy in new work and for the revision of the arithmetic procedures (Board of Education, 1937; Cochran, 1960; Mutch, 1916). However, such a focus possibly contributed to the belief that mental arithmetic entailed the presentation of a series of

often one-step examples, the focus of which was the gaining of correct answers, the traditional view of calculating mentally.

It is with respect to the role of mental arithmetic as a means for ‘improving the tenacity of the mind’ (Wilkins, 1886, p. 40) that sharp distinctions may be drawn between past beliefs and those currently advocated by mathematics educators, although aspects of this traditional view appear to remain in the minds of many Queensland teachers and administrators (Morgan, 1999). Such a view has as its origins the tenets of formal discipline that were espoused particularly during the 19th century. This theory held that the mind was composed of a number of distinct powers or faculties, including memory, attention, observation, reasoning and will. During the late nineteenth century, Robinson (1882) had maintained that the value of giving complex calculations, beyond the requirements of the various syllabuses of the period, was ‘the formation of a power of concentrating all the faculties on the performances of an allotted task ... [so that] the mind ... [would] prove capable of any amount of labour upon other tasks’ (p. 178). Such a belief resulted in mental arithmetic being considered essentially as the ‘working [of] certain hard numbers in the shortest time by the shortest method’ (‘Mental Arithmetic: A Few Suggestions,’ 1910, p. 176).

Despite the discrediting of these beliefs early in the 20th century, and statements by senior Departmental officers affirming their belief that there is no such thing as general mental training, and that learning in one subject cannot be transferred to another (Edwards, 1936), teachers and inspectors retained their beliefs in the role of mental arithmetic as a means for quickening the intelligence, developing judgement, improving reasoning (Baker, 1929; Mutch, 1924), and for enhancing an individual’s ability to concentrate on mathematical tasks (Martin, 1916). The maintenance of these beliefs was supported by the influence of the *English* view of arithmetic, which represented arithmetic as logic (Ballard, 1928). Hence, mental arithmetic was judged to be the means by which children were trained to think and reason: ‘Intelligence in Arithmetic should be secured through the medium of mental exercises’ (Bevington, 1925, p. 83), with accuracy in thinking and reasoning of paramount importance (Martin, 1920). Nevertheless, Burns (1973) concluded that most teachers were primarily concerned with imparting factual knowledge. Hence, their concern was almost exclusively with memory.

The nature of mental arithmetic

Essential to the nature of mental computation, as now conceived, is the notion of mental strategies as being fleeting, variable, flexible, active, holistic, constructive, and iconic, strategies that are not necessarily designed for recording, and require an understanding of the mathematical relationships embodied in the problem task environment. These features contrast with those of the standard paper-and-pencil algorithms. Such procedures are standardised, contracted, efficient, automatic, symbolic, general, and analytic. Further, they are not easily internalised and encourage cognitive passivity (Plunkett, 1979).

As early as 1887, District Inspector Kennedy had argued against mental arithmetic being viewed as the mere application of fixed rules (Kennedy, 1887), rules that were

generally listed at the end of the treatises on mathematics used by teachers. Park (1879) had previously stressed that where rules were to be taught they should not be ‘got by rote’ (p. 43). Nonetheless, the period 1860–1965 was characterised by an importance placed upon the rote application of short methods of mental calculation, although it was only in the 1930 Syllabus, and its amendments in 1938 and 1948, that specific references were made to such calculations in schedules and syllabuses. That the 1952 Syllabus did not refer to such methods was a situation with which District Inspector Crampton (1956) did not approve, maintaining that practical short methods should have been prescribed.

Little emphasis appears to have been placed on permitting children to devise their own mental strategies. Nevertheless, teachers were occasionally encouraged in articles in the *Queensland Teachers’ Journal* and the *Education Office Gazette* to permit children to invent short methods for themselves. Some encouragement was also contained in the 1930 Syllabus and the 1948 Amendments — complementing practice in the use of short methods, Grade V children were to be encouraged to devise different solutions for particular problems, but only ‘after the pupils [had] been thoroughly exercised in any rule’ (Department of Public Instruction, 1930, p. 41; 1948, p. 13).

The 1952 Syllabus appears somewhat equivocal about the nature of mental arithmetic. While stating that ‘its application should not be limited to short methods or similar devices,’ it also maintained that ‘the processes applied orally [that is, mentally,] are the same as those used in written operations’ (Department of Public Instruction, 1952, p. 2). It is the latter view of mental arithmetic that was emphasised for all grades in all schedules and syllabuses from that of 1904, albeit a view that was not often realised, as recorded by District Inspectors in their reports (Morgan, 1999). Teachers tended to view mental arithmetic as the presentation of a series of often unrelated oral questions, the aim of which was to obtain correct answers, speedily and accurately.

Such a conviction was supported by the format of textbooks and articles on mental arithmetic in the *Queensland Teachers’ Journal*. As Lidgate (1954) noted, ‘There [was] a general tendency to test rather than to teach Mental’ (p. 2). This, combined with the practice of providing examples beyond the requirements of particular grades by teachers, District Inspectors and Head Teachers, eventuated in Greenhalgh (1947) condemning the way in which mental arithmetic was taught for the nervous strain that was being placed on children. However, this was a practice that continued at least into the late-1950s. District Inspector Kehoe (1957) recorded in his 1956 Annual Report that ‘some teachers were puzzling their pupils in tables and giving them mental gymnastics by ingenious and complicated methods they have evolved for working, for example, extensions tables and measurement tables’ (p. 2).

Approaches to teaching mental arithmetic

Two broad approaches to teaching mental computation — a behaviourist approach and a constructivist approach — have been suggested by Reys, Reys, Nohda, and Emori (1995). The former views mental computation as a basic skill and is considered to be an essential prerequisite to written computation, with proficiency gained through direct teaching. However, the constructivist approach contends that mental computation is a

process of higher-order thinking in which the act of generating and applying mental strategies is significant for an individual's mathematical development. In both approaches, the emphasis is placed on mental strategies.

This contrasts with the traditional approach to calculating mentally. Although also behaviourist in nature, it focusses on the speed and accuracy with which answers are obtained. Syllabus development was guided by an instrumental (Selleck, 1968) view of education that held that arithmetic was a tool, with computational skills all important, rather than understanding — the importance for pupils was for them 'to become good practical calculators; for deficiency in this, no amount of arithmetical ingenuity or theoretical knowledge will compensate' (Joyce, 1881, p. 203).

Teachers were impelled by District Inspectors and syllabus documents to make mental arithmetic a part of every lesson. The aim was for it not to be taught in isolation (Department of Public Instruction, 1914), but as an introduction to all written work_for teaching the principle of an operation, for promoting speed and accuracy at the mechanical stage, and for applying the particular operation in problem situations. Hence, mental examples were to be based on the written work that was to follow, with these carefully graded prior to a lesson. Although mathematics educators would agree that mental computation needs to be a part of every mathematics lesson, it should not be considered as a separate topic with a set of rigidly ordered skills. Rather, a focus on mental calculation should receive an on-going emphasis throughout all situations requiring computation, thus leading children to view mental methods as legitimate computational alternatives.

Calculating mentally can provide numerous opportunities for the development of mathematical thinking. However, it requires regular and systematic practice. District Inspectors of Schools prior to 1965 often exhorted teachers to provide additional opportunities for practice in mental arithmetic. Preferably this was to be undertaken for short periods in the mornings when children's minds were fresh (Baker, 1929; Drain, 1941; 'Teaching Hints,' 1908). Lessons were characterised by rapid question delivery, with the profit generally considered to be in the number of questions answered correctly (Gladman, 1904). Martin (1916) advocated that explanations should be kept to a minimum as a means for ensuring a 'concentrating of the mind' (p. 135). This contrasts with current beliefs about teaching mental computation, the essence of which is the focus on assisting children to see *how* to calculate mentally.

Except for rare commendations, District Inspectors of Schools were consistently highly critical of the standard of mental calculation and of the teaching methods used during the period 1860–1965. It was occasionally recognised that mental arithmetic made greater demands on a teacher's time, energy, and ability than written work (Canny, 1910), particularly in small schools where multiple grades had to be taught. Nonetheless, teachers were often criticised for not being able to produce properly sequenced lists of mental examples, for an overuse of the departmental written arithmetic cards, and for not attempting any 'individual diagnosis to discover causes and failures, [with] the backward pupils hardly ever called upon to do any of the work' (Farrell, 1929, p. 290). These criticisms were frequently a consequence of the large classes taught, often artificially enlarged by reducing the number of pupils in the Scholarship class.

Also mitigating against a teacher's ability to adequately develop mental arithmetic skills were the attempts to impose syllabus change unilaterally, without significant professional development. This, in addition to the generally poor quality of teacher training for those who came through the pupil-teacher system, or no training at all for many teachers employed in Provisional Schools before 1909. The traditional beliefs and practices espoused by senior teachers therefore tended to be perpetuated.

The authoritarian atmosphere that characterised classrooms of this period also contributed to the uncompromising nature of mental arithmetic. 'The accumulated experience of teachers in the system led them to believe that strict order, the threat of sanctions, repetition, drill and cramming were likely to achieve results in examinations and during the inspector's visit. (England, 1971, p. 193). A premium was placed on knowledge and learning by rote, and the use of examples that often exceeded syllabus requirements, emphases that do not contribute to the development of idiosyncratic and flexible mental strategies.

Conclusion

The insights presented above are based on evidence that is substantially limited to the beliefs and opinions of two groups of stakeholders, namely, those of District Inspectors of Schools, as recorded in their annual reports, and of teachers, as expressed through publications of the Queensland Teachers' Union. Although these two groups of departmental officers often expressed differing opinions about issues that impinged on classroom practice, taken together, their recorded views, in conjunction with syllabus documents, have enabled a clear picture of mental arithmetic to emerge.

Mental arithmetic retained its place in the syllabuses taught in Queensland from 1860 to 1965 for two key reasons. First, despite the doubt cast on the validity of the theories of formal discipline, Queensland teachers and District Inspectors, generally, retained their belief in mental arithmetic as a means for training the mind to logical and critical thought. Second, in common with current beliefs, its place was maintained because of its recognised social usefulness. This aspect gained ascendancy from the 1930s when the predominant educational philosophy was realism (Greenhalgh, 1947), coupled with the belief that the only things worth teaching were those that had some obvious and immediate use.

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The development of students' thinking in chance and data*

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Increased emphasis on chance and data in the mathematics curriculum calls for teachers to have a greater understanding of the development of students' thinking in probability and statistics. Recent research with students and teachers has facilitated the development of frameworks for students' thinking in both domains. The purposes of the frameworks are to describe students' thinking in terms of levels across the concepts in chance and data, and to inform teachers' planning of suitable learning activities. This paper outlines two such frameworks developed by Jones et al. (1997, 2001) one for statistical thinking and one for probabilistic reasoning and describes their implications for teaching and planning of instruction.

Introduction

The teaching of probability and statistics in primary schools has come a long way in the last 60 years, if the Queensland experience is anything to go by. A perusal of mathematics papers of the State Scholarship Examination from the 1940s and 1950s (Department of Public Instruction, Queensland) reveal no questions on probability and statistics, indicating that these topics would not have been taught in state primary schools in that era. However, by the 1970s, the syllabus had been amended a couple of times, firstly in 1966 to cater for the introduction of decimal currency in Australia and to incorporate ideas of the *New Mathematics* which included set theory and to some extent statistics. It was amended again in 1975 to reflect Australia's conversion from imperial to metric measurement in the early 1970s.

The 1975 syllabus (Department of Education, Queensland, 1975) contained one strand on *Statistics and Graphs* which dealt with representing data with graphs — starting in Year 2 with pictographs, and moving on to bar graphs in Year 4, circle graphs in Year 5, line graphs in Year 6, and histograms and frequency polygons in Year 7. Measures of central tendency were included in Years 6 and 7. The only mention of probability was a suggestion of a small number of investigations of random events at the end of the statistics section for Year 7.

* This paper has been subject to peer review.

By the late 1980s the Syllabus (Department of Education, Queensland, 1987) had expanded the emphasis on probability and statistics with specific mention of analysing chance situations, identifying outcomes, calculating numerical probability, collecting data, organising data, representing data, interpreting data, and investigating problems involving data. This syllabus is still current while a new one is being trialled (Queensland School Curriculum Council [QSCC], 2002). A significant event in the development of mathematics curricula in Australia was the publication of *A National Statement on Mathematics in Australian Schools* (Australian Education Council, 1991) with its organisation of the curriculum into five strands, one being *Chance & Data*. The *Statement* highlighted the importance of this strand, and stated that ‘a grasp of concepts in chance, data handling and statistical inference is critical for the levels of numeracy appropriate for informed participation in society’.

The Queensland trial syllabus in (QSCC, 2002) describes the content in *Chance* in terms of using the language of chance, identifying outcomes, determining and interpreting numerical probabilities. The syllabus content for *Data* is described in terms of collecting, interpreting, and communicating data, data displays, sampling and data analysis.

Learning frameworks

Probability and statistics were ‘late starters’ in the school mathematics curriculum compared to other topics like number and geometry, and hence less attention has been paid to them in terms of investigating how students learn their concepts and processes, and how teachers can assist students build valid knowledge structures and use them to solve problems. Much research has been conducted into students’ learning of number and geometry and ways that teachers can facilitate this learning. Various frameworks for the learning of number and geometry have been available for teachers and curriculum writers for some years now (for arithmetic see Steffe, Cobb & von Glasersfeld, 1988, and for geometry see van Hiele & van Hiele-Geldof, 1958). Such frameworks describe the constructs of the topic, along with the developmental stages that students pass through with instruction, maturation and experience.

A framework is a blend of content and cognition and, as such, is relevant to classroom teachers, learning-support staff, curriculum writers, and textbook writers. One could argue that a knowledge of learning frameworks in school mathematics is not just relevant but essential for such personnel. According to Shulman’s (1986) position on teacher knowledge, teachers need to have content knowledge (e.g. a knowledge of the structure of the subject), and pedagogical content knowledge (e.g. understanding the conceptions and pre-conceptions of students of different ages, ordering of content, and how to re-organise the understanding of learners). Hence, teachers can use a framework to assess the levels of understanding of students in key concepts, and use those data to inform their teaching and planning. Curriculum writers and textbook writers can use frameworks to assist in structuring and ordering learning sequences. Frameworks for probability and statistics comparable to those for number and geometry are more recent additions to the mathematics education literature.

The statistical-thinking framework

The statistical-thinking framework described in this paper (Jones et al., 2000) was developed by Jones and his colleagues at Illinois State University after extensive research, development and validation with elementary school pupils. The framework (Table 1) consists of four constructs (Describing Data Displays, Organising and Reducing data, Representing Data, & Analysing and Interpreting data), which are described at four levels (Idiosyncratic, Transitional, Quantitative & Analytical).

The construct *Describing Data Displays* relates to a student's reading and describing displays such as graphs and tables, understanding the conventions used in the displays, recognising correspondences between two different displays of the same data, and being able to evaluate the effectiveness of different displays. *Organising and Reducing Data* refers to grouping and ordering data, being able to explain the basis for the organisation of the data, recognising when data reduction occurs, and being able to calculate measures of typicality and spread. *Representing Data* relates to being able to complete or produce displays of data, i.e. various types of graphs, and includes the ability to produce a display which shows some reorganisation of the data. *Analysing and Interpreting Data* refers to being able to make comparisons, reading between the data, making inferences with the data (i.e. reading beyond the data), and recognising what a display does not say about the data.

In the framework, the term 'levels' is used to describe a particular sequence of stages that a student's thinking passes through over time. Level 2 should be seen to subsume Level 1 and be more sophisticated in nature. Similarly, Level 3 should be seen to subsume Level 2 and be more sophisticated in nature, and so on to Level 4. The levels were based on, and hence have similarities with, the developmental model of Biggs & Collis (1991) which sets out four cognitive levels through which students progress sequentially in their learning of concepts (pre-structural, uni-structural, multi-structural, and relational).

Applications and implications of the statistical-thinking framework

There are a number of applications and implications that arise from a knowledge and understanding of the statistical-thinking framework. Four of these worth consideration are teachers' assessment of students' strengths and weaknesses in data handling, teachers' planning of lessons and units in statistics, curriculum and textbook writers design of learning sequences, and researchers' investigations of students' learning of statistical concepts.

Table 1. Statistical-thinking framework (Jones et al., 2000).

Construct / Level	Level 1 Idiosyncratic	Level 2 Transitional	Level 3 Quantitative	Level 4 Analytical
Describing Data Displays	<ul style="list-style-type: none"> demonstrates little awareness of the cosmetic features (e.g. title, axis labels) of the display recognises when different displays represent the same data by giving a justification based on subjective judgements considers irrelevant or subjective features when evaluating the effectiveness of different displays of the same data set 	<ul style="list-style-type: none"> demonstrates some awareness of the cosmetic features (e.g. title, axis labels) of the display recognises when different displays represent the same data by giving a justification based on cosmetic features (e.g., title, axis labels) of the displays focuses only on one aspect when evaluating the effectiveness of different displays of the same data set 	<ul style="list-style-type: none"> generally demonstrates complete awareness of the cosmetic features (e.g., title, axis labels) of the display recognises when different displays represent the same data by establishing partial relationships between the displays focuses on more than one aspect when evaluating the effectiveness of different displays of the same data set 	<ul style="list-style-type: none"> demonstrates complete awareness of the cosmetic features (e.g. title, axis labels) of the display recognises when different displays represent the same data by establishing precise numerical relationships between the displays provides a coherent and comprehensive explanation when evaluating the effectiveness of different displays of the same data set
Organising and Reducing Data	<ul style="list-style-type: none"> does not attempt to group data into classes does not attempt to order data is not able to describe data in terms of representativeness or 'typicality' is not able to describe data in terms of spread 	<ul style="list-style-type: none"> groups data into classes on the basis of criteria which the student may not be able to explain orders data but may not be able to explain the value of doing this describes representativeness of data using invented measures which are partially valid describes spread of data using invented measures which are partially valid 	<ul style="list-style-type: none"> groups data into classes and can explain the basis for this grouping orders data and is able to explain the value of doing this describes representativeness of data using the mode or invented measures which are valid describes spread of data using invented measures which are valid 	<ul style="list-style-type: none"> can group the data into classes in more than one way and can explain the bases for these groupings can order the data in different ways and can explain the bases for these different orderings describes representativeness of data in terms of measures of centre such as median or mean describes spread of data in terms of a common measure such as the range

Construct / Level	Level 1 Idiosyncratic	Level 2 Transitional	Level 3 Quantitative	Level 4 Analytical
Representing Data	<ul style="list-style-type: none"> • constructs an idiosyncratic or invalid display when asked to complete a partially constructed graph associated with a given data • produces an idiosyncratic display that does not represent the data 	<ul style="list-style-type: none"> • constructs a display that is valid in some aspects when asked to complete a partially constructed graph associated with a given data set • produces a display that represents the data but does not attempt to reorganise the data 	<ul style="list-style-type: none"> • constructs a display that is valid when asked to complete a partially constructed graph associated with a given data set • produces a display that shows some attempt to reorganise the data 	<ul style="list-style-type: none"> • constructs a valid display and justifies the procedure when asked to complete a partially constructed graph associated with a given data set • produces multiple valid displays some of which reorganise the data
Analysing and Interpreting Data	<ul style="list-style-type: none"> • reads literally from the display ('reading the data') • makes no response or an irrelevant response to the question, 'What does the display <u>not</u> say about the data?' • expresses idiosyncratic notions about who would use the data 	<ul style="list-style-type: none"> • compares data within the display or data set ('reading between the data') • makes a relevant but limited response to the question, 'What does the display <u>not</u> say about the data?' • expresses limited notions about who would use the data 	<ul style="list-style-type: none"> • makes inferences concerning the variables beyond the scope of the data present ('reading beyond the data') • makes multiple relevant responses to the question, 'What does the display <u>not</u> say about the data?' • expresses meaningful but imprecise notions about who would use the data 	<ul style="list-style-type: none"> • makes inferences (about two or more data sets or displays) that involve reading between or beyond the data • makes a comprehensive contextual response to the question, 'What does the display <u>not</u> say about the data?' • expresses insightful and precise notions about who would use the data

Firstly, a teacher can use the framework to assess students' strengths and weaknesses in data handling, and establish levels of thinking for each of the constructs. This would be done by asking questions based on the constituent elements of the framework. For example, to assess students' level of thinking for the construct *Describing data displays*, a teacher could ask them to explain what the connection is between a table of data and its graphical representation. Then for *Organising and reducing data*, a teacher could ask students to take a set of raw data and organise it according to some criterion, and then explain the basis of the organisation. Similarly, for *Representing data*, the task could be to draw a graphical representation of a data set in more than one way. To assess students' level of thinking for the construct *Analysing and interpreting data*, a suitable task would be to make comparisons between elements of a graph (e.g. which is the greatest amount, or when was it least?). Students' responses to such questions and tasks would then be assessed by the teacher according to the specific elements of the framework (the dot points listed in Table 1) determining which levels the responses correspond to.

Secondly, teachers can use the framework to ensure that their teaching of statistics has a balance of coverage across the constructs. An analysis of results of tests based on the framework would identify for teachers the areas students understand (and the level of understanding), and the areas which cause difficulties and hence require further attention. The analysis of results would also show the range of levels demonstrated by the students in the one class. Further, the four constructs give teachers an indication of what the big ideas and major purposes of the teaching of data handling are. Their planning of data lessons should ensure that each of the constructs are covered adequately. It should be noted that the process of *collecting data* should also be included in teaching of data handling as a prelude to *organising and reducing data*. The framework does not refer to *collecting data* (which I see as an omission), but I believe that it also is an important element of the data handling process especially when it is likely that students will be more motivated to analyse data they have collected themselves than second-hand data. An integrating idea is to have students engage in projects in which they design a study of a relevant issue, devise the data collection instruments, collect the data, analyse the data, represent and interpret the data, draw conclusions and make inferences.

Thirdly, similar applications exist for curriculum writers in terms of placing emphasis on the big ideas of statistics, ensuring that the curriculum contains a balance of coverage across the constructs, recognising a developmental sequence of levels of understanding and performance, and promoting the use of problem-based projects to integrate the components of the framework in a meaningful, purposeful and motivating way.

Lastly, for researchers investigating students' statistical thinking, the framework offers a guide to designing a study, a choice of parameters for the study, dimensions on which to measure changes in students' thinking both qualitatively and quantitatively, and a structure for evaluating the effectiveness of teaching approaches. For example, the framework used to assist in the design and evaluation of a teaching experiment by Jones et al. (2001) from which a number of conclusions were reached about students' statistical thinking. With respect to *describing data*, experiences with different kinds of

data seemed to focus children's thinking and produce descriptions which were less idiosyncratic. Children's thinking in *organising and reducing data* was problematic in that the children were reluctant to use paper and pencil to reorganise data, but technology proved helpful in stimulating their strategies. The instruction with technology and incomplete graphs stimulated children's sorting schema and their capability for constructing representations. With respect to *analysing and interpreting data* the students became more quantitative and precise in their responses.

In other research (Nisbet, 2001, 2002a, 2002b; Nisbet, Jones, Thornton, Langrall & Mooney, in press) it has been found that:

- primary, secondary and tertiary students have more difficulty with organising and representing *numerical* data compared to *categorical* data;
- size of sample is a factor in representing data in that students are more likely to represent large numerical data sets in an organised way than small data sets;
- mathematically-able students are more likely to organise numerical data validly than their less able counterparts, and
- teacher's prompts to organise data are effective in assisting students produce valid organised representations.

The conclusions from such research provide much information which is useful for teachers in their planning and instruction, as well as for curriculum and textbook writers in their goal setting, structuring of content, and sequencing of learning activities.

The probabilistic-reasoning framework

The probabilistic-reasoning framework described in this paper (Jones, Langrall, Thornton & Mogill, 1997) was also developed by Jones and his colleagues at Illinois State University after extensive research, development and validation with elementary school pupils. The framework (Table 2) consists of six constructs (Sample Space, Experimental Probability, Theoretical Probability, Probability Comparisons, Conditional Probability, & Independence), which are described at four levels (Level 1: Subjective, Level 2: Transitional, Level 3: Informal Quantitative & Level 4: Numerical).

The construct *Sample Space* refers to listing the outcomes of one- and two-stage experiments. The constructs *Experimental Probability* and *Theoretical Probability* involve being able to distinguish between the two ideas, understanding the role of sampling in determining numerical values, and applying these ideas to predict least likely and most likely events in one- and two-stage experiments. *Probability Comparisons* involve reasoning to distinguish 'fair' and 'unfair' probability situations. The construct *Conditional Probability* relates to recognising how the probability values differ in replacement and non-replacement situations. The construct *Independence* refers to being able to distinguish between independent and non-independent events, and understanding the independence of consecutive trials in an experiment.

Table 2: Probabilistic reasoning framework (Jones, Langrall, Thornton, & Mogill, 1997)

Construct	Level 1 Subjective	Level 2 Transitional	Level 3 Informal Quantitative	Level 4 Numerical
Sample Space	<ul style="list-style-type: none"> Lists an incomplete set of outcomes for a one-stage experiment 	<ul style="list-style-type: none"> Lists a complete set of outcomes for a one-stage experiment and <i>sometimes</i> for a two-stage experiment 	<ul style="list-style-type: none"> Consistently lists the outcomes of a two-stage experiment using a partially generative strategy 	<ul style="list-style-type: none"> Adopts and applies a generative strategy to provide a complete listing of the outcomes for two- and three-stage cases
Experimental Probability of an Event	<ul style="list-style-type: none"> Regards data from random experiments as irrelevant and uses subjective judgments to determine the most or least likely event Indicates little or no awareness of any relationship between experimental and theoretical probabilities 	<ul style="list-style-type: none"> Puts too much faith in small samples of experimental data when determining the most or least likely event; believes that any sample should be <i>representative</i> of the parent population May revert to subjective judgments when experimental data conflict with preconceived notions 	<ul style="list-style-type: none"> Begins to recognise that more extensive sampling is needed for determining the event that is most or least likely Recognises when a sample of trials produces an experimental probability that is markedly different from the theoretical probability 	<ul style="list-style-type: none"> Collects appropriate data to determine a numerical value for the experimental probability Recognises that the experimental probability determined from a large sample of trials approximates the theoretical probability Can identify situations in which the probability of an event can be determined only experimentally
Theoretical Probability of an Event	<ul style="list-style-type: none"> Predicts most or least likely event on the basis of subjective judgments Recognises certain and impossible events 	<ul style="list-style-type: none"> Predicts most or least likely event on the basis of quantitative judgments but may revert to subjective judgments 	<ul style="list-style-type: none"> Predicts most or least likely events on the basis of quantitative judgments Uses numbers informally to compare probabilities 	<ul style="list-style-type: none"> Predicts most or least likely events for one- and simple two-stage experiments Assigns a numerical probability to an event (a real probability or a form of odds)
Probability Comparisons	<ul style="list-style-type: none"> Uses subjective judgments to compare the probabilities of an event in two different sample spaces. Cannot distinguish 'fair' situations from 'unfair' ones. 	<ul style="list-style-type: none"> Makes probability comparisons on the basis of quantitative Begins to distinguish 'fair' situations from 'unfair' ones. 	<ul style="list-style-type: none"> Uses valid quantitative reasoning to explain comparisons and invents own way of expressing the probabilities Uses quantitative reasoning to distinguish 'fair' and 'unfair' probability situations 	<ul style="list-style-type: none"> Assigns a numerical probability and makes a valid comparison

Construct	Level 1 Subjective	Level 2 Transitional	Level 3 Informal Quantitative	Level 4 Numerical
Conditional Probability	<ul style="list-style-type: none"> Following one trial of a one-stage experiment does not always give a complete listing of possible outcomes for the second trial Uses subjective reasoning in interpreting with and without replacement situations 	<ul style="list-style-type: none"> Recognises that the probability of <i>some</i> events changes in a without replacement situation; however, recognition is incomplete and is usually restricted only to events that have previously occurred 	<ul style="list-style-type: none"> Recognises that the probability of all events changes in a without replacement situation Can quantify changing probabilities in a without replacement situation 	<ul style="list-style-type: none"> Assigns numerical probabilities in with replacement and without replacement situations Uses numerical reasoning to compare the probability of events before and after each trial in with replacement and without
Independence	<ul style="list-style-type: none"> Has a predisposition to consider that consecutive events are always related Has a pervasive belief that one can control the outcome of an experiment 	<ul style="list-style-type: none"> Begins to recognise that consecutive events may be related or unrelated Uses the <i>distribution</i> of outcomes from previous trials to predict the next outcome (representativeness) 	<ul style="list-style-type: none"> Can differentiate independent and dependent events in with and without replacement situations May revert to strategies based on representativeness 	<ul style="list-style-type: none"> Uses numerical probabilities to distinguish independent and dependent events

Applications and implications of the probabilistic-reasoning framework

Again there are a number of applications and implications that arise from a knowledge and understanding of the probabilistic reasoning framework. Four of these worth consideration are teachers' assessment of students' strengths and weaknesses in probabilistic reasoning, teachers' planning of lessons and units in probability, curriculum and textbook writers design of learning sequences in chance, and researchers' investigations of students' learning of chance concepts.

Firstly, a teacher can use the framework to assess students' strengths and weaknesses in probability, and establish levels of thinking for each of the constructs. This would be done by asking questions or setting tasks based on the constituent elements of the framework. Of course, the level of questioning would be determined by the age/grade of the students, and the teacher's background knowledge of the students. For example, to assess students' level of reasoning for the construct *Sample Space*, a teacher could ask them to describe all the possible outcomes for the rolling of two dice, and see how systematically they do it. Then for *Experimental Probability*, a teacher could ask students to explain why, when throwing a coin 10 times, it is possible for a *head* to occur 8 times. Similarly, for *Theoretical Probability*, the task could be to explain what the expected number of heads would be for 1000 throws. To assess students' level of thinking for the construct *Probability Comparisons*, a suitable task would be to explain why a two-player game involving rolling two dice and finding differences is unfair if one player wins when the difference is 0, 1 or 2, and the other player wins when the difference is 3, 4, or 5. Students' responses to such questions and tasks would then be assessed by the teacher according to the specific elements of the framework (the dot points listed in Table 2) to determine which levels the responses corresponds to. For the construct *Conditional Probability*, a teacher could set the context of selecting cards from a regular pack of playing cards, and then ask the student to calculate the probability of selecting a heart a second time after a heart had been selected first (with no replacement).

Secondly, as with the statistical-thinking framework, teachers can use the probabilistic-reasoning framework to ensure that their teaching of probability has a balance of coverage across the constructs. An analysis of results of tests based on the framework would identify for teachers the areas students understand (and the level of understanding) and the areas which cause difficulties, and hence require more attention. An analysis of results would also show the range of levels demonstrated by the students in the one class. Further, the six constructs give an indication to teachers of what the big ideas and major purposes of the teaching of probability in schools are. Their planning of chance lessons should ensure that each of the constructs are covered adequately.

Thirdly, there are similar applications for curriculum writers in terms of placing emphasis on the big ideas of probability, ensuring that the curriculum contains a balance of coverage across the constructs, recognising a developmental sequence of

levels of performance, and promoting activities which integrate the components of the framework in a meaningful and purposeful way.

Lastly, for researchers investigating students' probabilistic reasoning, the framework offers a guide to designing a study, a choice of parameters for the study, dimensions on which to measure changes in students' reasoning both qualitatively and quantitatively, and a structure for evaluating the effectiveness of teaching approaches.

It should be recognised that the two frameworks described in this paper have been developed and validated on the evidence provided through research with students in classrooms. However, as with all theories, the frameworks are not set in concrete, but are subject to revision on the basis of further research evidence. Nevertheless, I believe that the frameworks can assist teachers and planners to focus on students thinking and reasoning with respect to the big ideas in probability and statistics, and hence will facilitate students' progress towards richer conceptions and enhanced performance in the areas of chance and data.

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Models and representations: Do they have a role in a conceptual understanding of multiplication?

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An important aspect of conceptual development in primary students is the development and use of models, symbols and representations. The paper looks at upper primary school students' understanding of multiplication by categorising their explanations of symbols and equations as predominantly procedural or conceptual. After a short teaching experiment focussing on models and representations, especially arrays and area models, there was a noticeable shift towards more conceptual explanations.

Findings from the Victorian Middle Years Numeracy Research Project (Siemon & Virgona, 2002) show that one of the major problem areas for students in Years 5–9 is multiplicative thinking. Research in the area of multiplication has identified many misconceptions, such as 'multiplication always makes bigger, division always makes smaller' (Bell, Fischbein & Greer, 1984), which could be viewed as contributing factors. These misconceptions have been attributed in by some researchers to the over-generalisations from whole numbers to decimals (Tirosh & Graeber, 1989). Another factor, identified by other researchers as a concern at this level, is over-reliance on processes and procedural understanding to the detriment of conceptual understanding (e.g. Verschaffel & de Corte, 1996). Wearne and Hiebert (1988a) suggest

middle school students' knowledge of mathematics appears to be more procedural than conceptual... symbols begin to live a life of their own; ... students manipulate the symbols by recalling and applying memorised rules... [and] the student who attempts to make sense of the... manipulations is something of an anomaly (p. 220).

It is in the transition years of *middle primary* school that the foundations for the middle school are built, and where early understandings of multiplication are 'reconceptualised' (Greer, 1994, p. 73) and further developed to deal with the increasing complexity of multiplication involving decimals and fractions. The focus of this paper, which is part of a larger study, is the exploration of Year 5 students' understanding of multiplication as indicated by their explanations of symbols and equations, and how these understandings can be viewed as either predominantly procedural or conceptual.

Graeber and Tirosh (1990) investigated the concepts of multiplication and division held by a group of primary students in Israel and the United States of America. This study draws on their work.

The following question will be addressed:

To what extent can the conceptual meaning of symbols and equations involving decimals be enhanced by teaching sessions focussed on making meaning through models and representations?

Method

Participants and procedure

As in the main study, the sample comprised fifteen Year 5 students, aged 10–12, five boys and ten girls. All students attended the same inner Melbourne primary school. The students had been instructed in basic multiplication from Year 2 and in decimals and fractions from Year 4.

The study was conducted as a teaching experiment. The main data was collected from interviews based on a protocol from a Graeber and Tirosh (1990) study. Data were gathered on understandings and conceptions that students already had about multiplication and decimals. Interviews, using the same questions and format, were conducted after the teaching sessions as well. At each interview the student was asked a series of questions relating to the four major areas to be covered. This paper is concerned with only two of these, terminology and definitions (Task 1) and facility with decimals (Task 7). Each interview was approximately 20–25 minutes. Responses were recorded, including attitudes and behaviour, and students' written responses were kept.

Results of Task 1 were initially analysed according to the students' understandings of multiplication, based on the Graeber and Tirosh (1990) categories, which focussed mainly on the connection with addition. The responses were then recategorised according to procedural or conceptual understanding, typified by either action or process based responses, or more static, generalised responses. The categories for analysis were as follows:

- Category A: Confusion with the symbol or with task requirement.
- Category B: Responses indicating procedural understanding.
- Category C: Responses indicating conceptual understanding.
- Category D: Responses indicating a more sophisticated conceptual understanding.

Preliminary studies had revealed that on a written test of multiplication problems, including examples with decimals, students were able to successfully select the multiplication as the correct operation only if the problem was familiar to them. The teaching program, an integral part of the main study, was devised to assist students in developing a broader understanding of multiplication and designed specifically to introduce and allow exploration of:

- (a) array and area or region models of multiplication;
- (b) discussion of the multiplication symbol and verbalisation;
- (c) models of multiplication situations where one factor is a whole number and one a decimal, and where both factors are decimals;
- (d) word problem representations (whole numbers by whole numbers and whole numbers by decimals); and
- (e) understanding of factor and product.

Of major interest to the author was how students were visualising or imaging a problem that was just presented in symbolic form. What meaning and understanding did they attach to it, or were they dependent on a procedural approach involving recall of a set of rules? At the initial interview the majority of students spent some time after doing an algorithm for $15 \times .6$, (Task 7a) trying to decide where the decimal point would go in the answer. Students were not able to bring to the task an understanding that would allow them to test their solution against what would be a reasonable result.

Results

The first section reports on the responses to Task 1 at the three interviews, in terms of categories A–D. The second section reports the responses to Tasks 7a and 7b, and then links these to the responses to task one.

Task 1 required the subjects to explain or describe what the multiplication sign or symbol meant. The symbol was presented in an equation $4 \times 5 = 20$.

Task 1

Show the student a card with the sentence $4 \times 5 = 20$.

What does this sign (point to \times) mean? What does multiplication mean?

Do you remember what we call these numbers (point to 4 and 5) in a multiplication sentence?

(Establish term 'factors'.)

What do we call this number (20)? (Establish term 'answer' or 'product'.)

The responses to the question, 'What does this sign mean?' have been grouped into the four main categories above. The number of responses in each of these categories at each interview is displayed in Table 1.

In the first group the symbol was misread as division. The subjects were searching for indications of the action that the symbol required them to perform; e.g. 'I think it's multiplication or divide, it means you have to put 4 into 5'. The second category of responses were those where the subjects expressed or implied a procedural understanding indicating an action or process. The responses were similar to instructions on how to 'do' multiplication and were characterised by the expression 'you have to'. A typical response was 'you have to times that number by that number (pointing to the 4 and the 5)'. The third category contained the responses that were more general. One such response was: 'It means this number that many times'

(indicating the 4 and 5 respectively). They were defined by the use of a static description or definition and made frequent reference to ‘groups of’. The final group was categorised by responses using more sophisticated language (factor and product) explaining the meaning of the symbol as a generalisation or definition, for example, ‘it means to increase the value of the first number by the second number’, and ‘this is the product and these are the factors’. Three of these subjects maintained their presence in this group across the three interviews.

Table 1. Number of responses of students in each category to Task 1 — Definitions.

Total students: 15			
Category	Interview 1	Interview 2 (one absent)	Interview 3
A: Confused	1	1	1
B: Procedural	10	3	3
C: Conceptual	4	6	8
D: Conceptual - sophisticated	0	4	3

Generally there was a change in responses from the first to the second and third interviews, showing a general shift from explaining multiplication in terms of a process or action to generalisations. Initially almost 67% of responses emphasised a procedural understanding and about 23% gave a response showing a more conceptual understanding. At the second interview, following the teaching sessions, 20% of the responses were procedural and almost 67% were conceptual (categories C and D combined). This was similar in the third interview. The percentage of conceptual responses rose from the first interview (23%) to the third interview (73% respectively).

Tasks 7a and 7b required the students to solve multiplication equations, each with a whole number and a decimal fraction as the factors.

Task 7a

Show student a card with the number sentence $15 \times .6 = ?$

Can you tell me what the hidden number is? Can you tell me anything about it?

Why do you think so?

Task 7b

Show student a card with the number sentence $.1 \times 10 = ?$

Can you tell me what the hidden number is? Can you tell me anything about it?

Why do you think so?

The results were grouped into three categories.

- Category 1: The students who were confused with decimals and unable to continue, and those students who attempted to use an algorithm to solve this problem, but were uncertain about procedure and gave an incorrect response were scored as zero.
- Category 2: The students who gave a correct response, but with no explanation, or by using rules to determine where to place the decimal point in the answer were scored as one.
- Category 3: The students who gave a correct response with a clear and meaningful explanation of the product being smaller than one of the factors ('The product is less than the bigger factor because the other factor is less than one.') were scored as two. Although not in the initial interview protocol, at interviews two and three the students were asked to respond to either $.1 \times 10$ or $15 \times .6$ with a pictorial representation or a story, or both.

Generally, in Task 7a there was a change in the responses from the first to the final interview. Initially, almost 50% of the responses were incorrect and a further 50% were correct and gave a rule as an explanation. There was a shift in the second and third interviews where 70% gave a correct response with clear explanations of the effect of multiplying by a decimal fraction. The responses to Task 7b were quite different where 40% gave a clear explanation at the first and second interviews, but at the third interview the majority of students gave a correct solution with no explanation.

Links between procedural and conceptual explanations of the multiplication symbol and responses to Tasks 7a and 7b

The results of the tasks involving decimals were compared with those of the Task 1 to compare the explanations of the symbols with the students' ability to explain and to solve multiplication equations involving decimals. These results are displayed in Table 2. For Task 1, categories A–D were rated with a score of 0–3. The results of Tasks 7a and 7b were allotted a score of 0, 1 or 2, for categories 1, 2 and 3 respectively, and the total was entered as a combined score for the two tasks.

Table 2 shows that the mean rose for Task 1 (definitions) and Tasks 7a and 7b (decimals) from the first to the second interview. Of interest is the drop in the mean at the third interview in Task 7, while the mean for Task 1 rose slightly at this interview. One student with a low score on Task 1 finished low and was unable to do the decimals at interviews 1 and 2, but succeeded by applying a rule at interview 3. Another student with a low score on Task 1 at the first interview, was correct on the decimal task, although not confident. At the second and third interviews her explanations had moved to the conceptual level and she succeeded at the decimal tasks. At the third interview she was able to explain that the second factor was a fraction, therefore the product would be lower than the first factor.

Overall, 9 students who shifted from a procedural explanation to a conceptual explanation were able to draw pictorial representations or generate a real world

problem to model either $15 \times .6 = 9$ or $.1 \times 10 = 1$, although some problems were without questions. Of the 50% who were unsuccessful in Task 7a at the first interview all gave a procedural explanation or were confused on Task 1. Three of these students gave a conceptual explanation at the second and third interviews, and were also successful on the decimal tasks.

Table 2. Scores of definition task and tasks with decimals

Student	Definitions			Decimals		
	Interview 1	Interview 2	Interview 3	Interview 1	Interview 2	Interview 3
Tt	1	0	0	0	0	1
Q	1	absent	2	0	absent	1
Ka	1	2	3	1	1	3
F	0	1	1	1	3	2
Te	1	2	2	1	3	3
Ta	1	1	2	1	2	3
P	1	1	1	1	3	2
B	2	3	3	3	3	3
O	1	3	3	3	4	3
Ki	1	2	2	3	4	3
Mc	1	2	2	1	4	3
E	1	1	2	4	4	3
Dh	2	2	2	3	4	3
L	2	3	3	3	4	3
D	1	3	2	1	3	3
Mean	1.1	1.8	2	1.7	3.0	2.6

Generally the scores for Tasks 1 and 7 were low at the first interview and rose at interview 2 following the intervention program.

Discussion and implications

The Graeber and Tirosh protocol (1990) was useful for understanding how students think about multiplication, allowing for explanations and discussion, in an accessible context. The sample, however, was small, and the question of how the teaching sessions, involving models and representations, impacted on the procedural and conceptual understanding of multiplication could not be resolved definitively without more tasks directly related to this aspect.

Results, however, did show that students tended to respond initially in a procedural way in their explanations of the symbols and in the decimal task, using only algorithms (largely unsuccessfully) to tackle the equations involving decimals. After the intervention sessions, however, they were able to make sense of the decimal tasks, relating them to real world stories and representations. The fact that twelve of the fifteen students were able to generate a story or draw a representation about the equations showed progression due to the intervention.

Constructing meaning in mathematics is a constant theme in research. Wearne and Hiebert (1988b) addressed this topic in a study about decimals with students in Grades 4, 5 and 6, that is relevant to this study. The degree of transfer from a known situation to a novel situation was recorded. Results showed that the performance on novel tasks was high when students applied a semantic process, but low when they applied routine procedures. This study showed similar results.

Experience with a variety of models and representations is an important aspect in the development of a more sophisticated understanding of multiplication. Verschaffel and De Corte (1996) discuss the inhibiting effect of rushing into formal operations which prevents students from developing ‘a rich and deep understanding’ (p. 111) of operations. A rich understanding has meaning, and with this meaning comes an ability to assess the reasonableness of a solution. In this study both procedural and conceptual understanding increased after teaching sessions focussed on meaning making.

Without meaning, the middle school number domain can, at best, result for children, in a memorised super-structure of symbolic manipulations that cannot easily be related to real-world situations and to which criteria of reasonableness cannot possibly apply (Fuson, 1988, p. 263).

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Challenging students' perceptions about learning in mathematics*

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Many students' approaches to learning mathematics are dependent on the teacher, and indicate a lack of awareness about learning strategies. To enhance learning, the following approaches were used: collaborative work, a focus on understanding, and reflection on learning. As these differed from most students' previous experiences of learning mathematics, some students resisted the changes. This paper discusses the students' responses as they were encouraged to accept greater responsibility for their learning.

Much has been written about lack of transfer of knowledge, surface or superficial learning, passive learning behaviours, and students' dependence on their teacher (Cohen, 1993; De Corte, 1995; White, 1996). A teaching approach to overcome this through the use of collaborative learning, focus on developing understanding, and reflection on learning, was implemented in 1997 and 1998 (Nothdurft, 2000). As this differed from the students' previous experiences in learning mathematics, the 1997 cohort resisted the change. They took time to adjust to a new regime, discovering 'that thinking is hard work, that taking responsibility and abandoning dependence is risky' (Black & Atkin, 1996, p. 90). They also had strong beliefs about what constituted appropriate learning and teaching behaviours and very restricted conceptions about what learning involved and what activities were likely to benefit them (Baird & Mitchell, 1991). The approach was much more successful the following year, as that cohort was involved in decisions about their learning and negotiated their role as students.

The research involved students in Year 12 Mathematics B at a small Catholic co-educational school in a regional Queensland city. Mathematics B is a pre-requisite subject for a number of university courses, and is generally chosen by students who have previously been successful in mathematics. Thus the students involved were keen to succeed, as they either needed the subject and/or expected to do well.

This paper focuses on the responses of the two cohorts of students to the changed teaching and learning approach.

* This paper has been subject to peer review.

Concerns about learning of mathematics

Many students believe that mathematics involves meaningless practice on routine exercises (Goos, Renshaw & Galbraith, 1998), emphasises memorisation and imitation (Brown, Stein & Forman, 1996), does not need to make sense (Schoenfeld, 1989), and does not require thought, only memorisation (Boaler, 1997). Such attitudes were found to be common among the students of this study.

At the beginning of each year, the students were asked about their goals for learning Mathematics B. These were mostly performance-focussed. The students then listed the strategies that they thought that they should use to learn mathematics better. The following were consistently nominated:

- work harder, including working all the time in class and doing all the homework;
- remember all the rules and formulae — ‘the whole point of maths is applying a rule to a situation and you need to know the rule’;
- do all the examples and exercises many times — ‘the only way to really do well in maths is to do problems over and over and over again and again’; and
- listen to the teacher.

These strategies indicate firstly that the students were dependent on the teacher. They regarded the teacher as the expert who ‘gives’ them the knowledge that they need, providing the only correct solution method to a problem and telling them what rules and algorithms they must remember. Secondly, the students lacked awareness of their learning. They had not reflected on how they learnt, and whether this was effective. As White et al. (1995, p. 474) stated: ‘learning is the core business of students. It is strange, though common, that few if any have any idea how to go about it’. Thirdly, because they were not concerned with how they made sense of what they were doing, they had difficulty in transferring their knowledge to other contexts. They did not make connections between different concepts and so were seldom successful in problem-solving.

The changed approach

The students mostly agreed that they should move from being passive recipients of knowledge towards accepting the responsibility for their own learning. This involved three main aspects. The first was a collaborative approach to learning so that the students would work together and help each other, rather than depend on the teacher for all help. The second was an emphasis on making sense of new concepts instead of just accepting and then memorising algorithms. The third was reflecting on how and what they were learning.

Collaborative learning

The aim was for all the students to be actively involved in doing mathematics:

- to discuss the meaning of concepts in their own language;
- to propose and argue about different solution strategies;

- to ask for clarification of other students' explanations; and
- to justify understandings and solutions.

I also wanted the students to ensure that each understood what the others were doing, to encourage all group members to participate fully, and to realise that they could understand better through explaining to others.

Unfortunately, this did not occur during much of the first year. As the students worked in groups in other subjects, I expected that they would be skilled in collaborative learning. However, the less able students remained dependent on me: not believing that they had sufficient expertise to solve problems without my help. About half of the groups seemed to regard the approach as legitimising talking in class. Only when I moved near them would they turn to the problems on which they were supposed to be working. While the more able students would willingly give solutions to their group, they were not always concerned whether the others understood their reasoning. Nor did they value discussing the advantages and disadvantages of different approaches and encouraging others to contribute.

I later realised that the change was too sudden, and the students were not adequately prepared for it. Instead of asking how they preferred to learn, I told them that I wished them to use this approach. The following year, the students choose to try the approach as it was proposed as a way of making learning more enjoyable and keeping each other on task, and of improving understanding through explaining one's own ideas and learning from each other. Group responsibilities and my expectations were made clear. Initially, the students were asked to take on specific roles. These roles included ensuring that the others explained their reasoning; writing up the group solution; and writing up the processes used to reach solution. They also worked in pairs: one would solve the problem and articulate the solution approach, while the other asked for clarification or for leads to the next step. Different groups presented their solutions to the class, with the other students questioning, adding to, or presenting alternatives to the solution. Most of the students were able to work well with other students, discussing what they were doing and helping each other by explaining their understandings. There was a sense that they were learning better through helping others. As one student said:

If someone in my group needs help, then I like to help them, because that helps me because I understand more what I'm doing. Like sometimes I know how to do it, but I don't understand why I'm doing it. But when I'm working with them, people ask questions, like 'Why are you doing that?' 'How are you doing that?' That makes me think about why I'm doing it.

Working meaningfully

While traditional practice involves presenting lots of information clearly, Hiebert et al. (1997) stated that less information is necessary; instead, more responsibility should be shifted to the students to search for or develop the information. These students' previous experience of school mathematics was that the teacher taught the content and they learnt this. They were quite comfortable about learning 'rules without reasons' (Skemp, 1976). They believed that good teaching meant giving clear comprehensible

procedures that they could learn. They practised and memorised what they were told to do with the aim of being able to retrieve enough to pass the exams. One student explained that he had managed previously by memorising the work, and asked, 'Do I really need to understand this? Can't I just learn it?'

The focus changed to their making sense of what they were doing. This occurred through an emphasis on understanding the rules and procedures that they used, on approaching new concepts as problems to be solved, and on their planning, monitoring and evaluating their learning. They were encouraged to take risks, to use different approaches, and to share these with other students. Strategies included asking the students to develop their own rules and procedures for different types of questions, introducing new concepts through solving related problems, focussing on the processes used to solve a problem, and doing fewer exercises, but exploring those more deeply. Understanding was regarded as the result of solving problems and using what they already knew, not something to be taught directly (Hiebert et al., 1997, p. 25).

During the first year, the middle third of the class felt that they were not being taught 'properly'. They wanted me to give them rules and examples and procedures to memorise. They said that learning would be easier if I would just tell them how to do the work: 'It's annoying how we have to work things out for ourselves — it's easier if we get told and then just apply it.'

While they agreed that it was valuable for them to understand what they were doing and why they were doing it, they still did not like this. The less confident students were concerned about their ability to make sense of things themselves without having the crutch of memorised procedures.

The approach was more successful the following year, as there was more discussion about learning and negotiation concerning teacher and student roles. Initially, the class and I discussed what learning meant. This resulted in their regarding an emphasis on understanding as the most sensible approach to learning. The focus was on their learning and the processes that they used, rather than on the content and 'correct' answers. This enabled them to develop an expectation that they could succeed. Most discovered that they did not need to memorise rules and procedures, and that it was instead easier to make sense of what they were doing. A student described the approach:

Teacher doesn't tell the answers or how to do it, but gives us hints and tells us to do it ourselves as much as possible. Teacher doesn't do the work for us, makes us do it ourselves.

Reflection

The other main strategy was to encourage the students to reflect on their understanding and behaviours, on self-regulatory learning strategies, and on their goals. The use of cognitive and metacognitive strategies has been found to contribute to achievement, to be important in the transfer of knowledge and application of problem-solving skills to novel situations (Cai, 1994; Garcia & Pintrich, 1992).

Reflection generally involved writing in journals or participating in whole class evaluation sessions. This occurred mostly with the 1998 cohort, and contributed to the increased awareness about what was involved in learning mathematics effectively, as well as enabling negotiation about teaching and learning strategies.

Journals were used to reflect on aspects of learning. Students wrote in response to prompts from me, for example, about what they had learnt in that lesson, how effectively they were using time in class, and their goals for the next month. They were also encouraged to write about any aspects of their learning, including successes and difficulties. While some appreciated this opportunity to communicate with me, not all students used their journals effectively as they were not used to writing in mathematics and some were not comfortable trying to articulate their thoughts.

In contrast, the evaluation sessions generally proved worthwhile for all. These occurred after the assessment of a topic, and the class would discuss teaching and learning strategies that they had found effective or otherwise, and might suggest alternatives. These public forums made individual students aware that they shared goals and concerns with others, and were not the only ones who had found some concept difficult or had never used a particular strategy previously. They also allowed the students input into what was occurring in their class. This contributed to the sense that we were all working together to enhance the learning of everyone.

Through reflection, most students developed an increased awareness of learning:

I have thought a lot more about learning and about how I am doing it. I use class time better. Last year, I would just sit and listen, but this year, I am trying to think about how it works and what to do with it... I found the reflection hard, because I've never had to think about what I do before — trying to recognise what I do and what I don't do in my learning.

Changing students' perceptions about learning

The different responses of the students in the two cohorts showed the importance of working with the students to change our joint practice compared with changing only my practice and expecting them to change as well. The approach was much more successful in the second year because of the students' involvement in their learning, through their awareness of the purpose of the change, and their having a forum in which to discuss aspects of the change and be able to influence it.

In the first year, I did not realise how important this was and so did not explicitly give those students such opportunities. Those students were expected to behave and use their time as I directed. I did not seek their ideas about whether any changes should be made. Because there was less structure and they were allowed more freedom, they wasted more time. About one third of the class displayed frustration and a lack of confidence. It was an approach to learning that took more effort, and it was very easy to slip back into the old comfortable ways of learning. Tensions also arose because of our differing expectations. By insisting on the students' taking responsibility of their own learning, and doing less direct teaching than I had previously, I was not behaving as they thought a 'good' teacher should.

In the second year, concerns about learning mathematics were discussed as a class. The changes were directly related to the students' goals and concerns. They were gradually introduced as a response to these concerns. Most of the students could see that the changed approach resulted in their increased confidence and success in, and greater enjoyment of, the work. This realisation did, however, require time. The class reflection in the evaluation sessions allowed the students to focus on what they were doing, what was working, and what they would prefer to be different. They knew that I sought their opinions and would be responsive to their concerns. This allowed the opportunity to resolve tensions, whereas in a traditional classroom, disaffected students would probably have disengaged silently. They learnt that learning was what they did themselves as they developed greater control over it. A changed understanding of the roles of teacher and students developed. As one student said:

I find because you're putting all the teaching on us, that we've got to understand it, we're accepting more responsibility, more than we did last year, because last year I kept wishing the teacher would do more. You just kept relying on the teacher, but now you realise now it's how much you do yourself. It makes you do more. You blame the teacher, but you realise now that you have to do it yourself as well.

Working collaboratively contributed to a changed understanding of learning. Most students realised that their learning was enhanced if they could work with others, and that explaining their understandings made them clearer to themselves. They realised the advantages of being both teacher and learner. This also contributed to a positive atmosphere in the classroom, as the students were supportive of each other and keen to work together.

Focussing on learning rather than content was different. It was important to develop an understanding of what it meant to learn: in particular, that it did not mean memorising algorithms and set problems, but being able to make sense of what one was doing and being able to apply one's knowledge in unfamiliar contexts. There was an explicit focus on learning strategies, as the strategies of planning, monitoring and evaluating were unfamiliar to most of the students. The problem-solving approach and reflection about understanding and learning strategies added to students' confidence, as they realised that they often could achieve more than they had previously expected.

The focus on reflection meant that the students became aware of their learning and of the changes in the approach to teaching and learning. In their private reflection, the students could admit their difficulties and concerns, set goals and later determine the extent to which these had been achieved, acknowledge to themselves their successes or failures, and consider why these had come about. In the public forum of the evaluation sessions, the students could focus on difficulties in learning mathematics. They could voice their concerns and work together to suggest solutions.

Conclusion

During the first year, I found teaching this way hard work. It seemed that I was in conflict with many of the students. Most of them would have preferred me to revert to my 'old' approach to teaching, telling them what they should be doing and then ensuring that they did it. I found that I needed to justify my approach to many of these

students, and reassure them that it would not disadvantage them. The whole class and I did not reach a shared understanding of what contributed to worthwhile learning, and there was no sense that we were working together as a community.

It was quite different the following year. By then, I had been using the new approach for a year, and so was more familiar with what I was trying to do. I felt more relaxed and was prepared to be more flexible and respond to the students' wishes more than I had the previous year. But the big difference was the attitude of the students. Because they were involved in decisions about their learning and were interested in how the changed approach was affecting it, they were prepared to accept responsibility for it.

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Assessing rational number knowledge in the middle years of schooling*

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There has been considerable research into the difficulties experienced by students learning fractions at all levels of schooling (see for example, Hunting, Davis & Pearn, 1996; Siemon & Stephens, 2001). In 2002 two forms of assessment were developed at the Catholic Education Office in Melbourne to determine the rational number knowledge of students from Years 5–8. Initially, a clinical interview was developed and subsequently two paper and pencil screening tests. The assessment instruments have been designed to probe the rational number knowledge of students in Years 5–8. In this paper, the process undertaken in the development of these three tests and the results from the initial trials will be discussed.

Introduction

Success In Numeracy Education [SINE] is the major numeracy approach being implemented in Victorian Catholic schools. SINE is designed to assist teachers to identify the mathematical understanding of the students they teach and to develop activities to help all students to progress at their relative level of understanding. Two components of SINE have been piloted and are now being implemented: SINE Prep to Year 4, and SINE Years 5 & 6. This paper focuses on work with fractions being undertaken as part of the pilot program for SINE Years 5–8.

Previous research

Over the past twenty years, research on rational number learning has focussed on the development of basic fraction concepts, including partitioning of a whole into fractional parts, naming of fractional parts, and order and equivalence. Kieren (1976) distinguished seven interpretations of rational number which were necessary to enable the learner to acquire sound rational number knowledge, but subsequently (Kieren, 1980; 1988) condensed these into five: whole-part relations, ratios, quotients, measures and operators. Kieren suggested that difficulties experienced by children solving rational number tasks arise because rational number ideas are sophisticated

* This paper has been subject to peer review.

and different from natural number ideas and that children have to develop the appropriate images, actions and language to precede the formal work with fractions, decimals and rational algebraic forms. Saenz-Ludlow (1994) maintained that students need to conceptualise fractions as quantities before being introduced to standard fractional symbolic computational algorithms. Streefland (1984) discussed the importance of students constructing their own understanding of fractions by constructing the procedures of the operations, rules and language of fractions.

Several researchers have noted how children's whole number schemes can interfere with their efforts to learn fractions (Behr, Wachsmuth, Post, & Lesh, 1984; Hunting, 1986; Streefland, 1984; Bezuk, 1988). Behr and Post (1988) suggested that children needed to be competent in the four operations of whole numbers, along with an understanding of measurement, to enable them to understand rational numbers. Others such as Siemon (2001) have drawn attention to the importance of multiplicative thinking and multiplicative strategies in supporting children's representations of fractions and their ability to work effectively with fractional quantities. All these authors note that rational numbers are the first set of numbers experienced by children that are not dependent on a counting algorithm. The required shift of thinking causes difficulty for many students.

Pothier and Sawada (1983) and Kieren (1976) have shown that students have a rich store of informal knowledge of partitioning and equivalence of fractions. Mack (1990) defined informal knowledge as 'applied, real-life circumstantial knowledge constructed by an individual student in response to problems posed in the context of real-life situations familiar to him or her' (p. 17). Mack concluded that students not only possessed informal knowledge, but that they were often able to connect this knowledge with formal symbols and procedures. Mack also found that where students possessed knowledge of rote procedures they focussed on symbolic manipulations.

Steffe and Olive (1990, 1993) showed that concepts and operations represented by children's natural language are used in their construction of fraction knowledge. Two distinct fraction schemes emerged from the research. In the iterative scheme, children established a unit fraction as part of a continuous but segmented unit. From this, children developed their own fraction knowledge by iterating unit fractions. The foundation of a measurement scheme occurred when the children's number sequence was modified to form a connected number sequence.

An Australian research project was designed to investigate the extent to which children's thinking processes might be associated with qualitative differences in their whole number knowledge when solving rational number tasks (Hunting, Davis, Pearn, 1996). Using the clinical method introduced by Opper (1977), twenty-eight Grade 3 children of average age 8 years 2 months from a predominantly middle-class government primary school were interviewed in March 1992. The interviews incorporated ten tasks designed to ascertain the children's understanding of both whole number and rational number concepts. Whole number tasks included counting composite units, verbal counting, and counting arrays. Rational number tasks included partitioning tasks, fraction tasks and a ratio task. This research highlighted the vast difference in the children's mathematical knowledge and the type of whole number strategies they used. The most successful students solving the whole number tasks,

were more successful and used superior strategies when solving rational number tasks. Students who relied on rules and procedures when solving whole number tasks were less successful with rational number tasks. They experienced some success with partitioning and ratio tasks but little or no success with fraction tasks set in various contexts used in the Hunting, Davis, Pearn research (1996). While most students were successful with tasks involving one-half, very few understood, or were successful, with tasks involving other unit fractions.

Fraction interviews

To assist teachers to assess the extent of their students' mathematical knowledge the authors decided it was imperative that teachers use an interview that would enable them to observe and interpret their students' actions as they worked on a set of tasks set in a variety of Fraction contexts. One purpose of clinical interviews in mathematics education is to characterise students' strategies, knowledge structures or competencies which yields information that is not easily accessible from other sources such as paper-and-pencil tests. The primary goal of interviewing is to determine the specific features and boundaries of a student's thinking. The process of thinking is considered more important than the correct solution. Student's verbal and non-verbal behaviour is observed, and from the observations the interviewer infers something about the student's internal representations, thought processes, problem-solving methods, or mathematical understandings.

Development of the fraction interview

A *fraction interview* (Pearn & Stephens, 2002) was developed to ascertain students' knowledge. The tasks included contexts such as discrete items, lengths, fraction walls, and number lines. To cater for the range of students in Years 5–8, tasks were designed with both a harder and easier question. For example in Task 4 of the fraction interview, the interviewer asked students to identify the whole when given the part (see Table 1). The initial task asked the task related to the fraction one-third. If students were successful they were asked about three-quarters and if unsuccessful students were asked to identify the whole if they were given the fraction one-quarter.

Table 1. Task 4 from the fraction interview (Pearn & Stephens, 2002).

Task 4	If student was successful:	If student unsuccessful:
Point to the picture of 4 lollies. <i>Say:</i> This is one-third of the lollies I started with. How many lollies did I start with? <i>Wait for student to respond.</i> <i>Say:</i> How did you decide?	Point to the picture of 6 lollies. <i>Say:</i> This is three-quarters of the lollies I started with. How many lollies did I start with? <i>Wait for student to respond.</i> <i>Say:</i> Why did you choose that many lollies?	Point to the picture of 2 lollies. <i>Say:</i> This is one-quarter of the lollies I started with. How many lollies did I start with? <i>Wait for student to respond.</i> <i>Say:</i> Why did you choose that many lollies?

The initial trial of the fraction interview

Sixty primary and secondary teachers participated in the pilot program of SINE Years 5–8. This pilot program was designed to focus on the hotspots identified by the Middle Years Research project conducted in Victoria (Siemon, 2001). In the first session of the pilot SINE program teachers received professional development which focussed on common difficulties with fractions experienced by students in the middle years of schooling. They also discussed ways to use the measurement model to develop understanding of fractions. This measurement model included the use of paper folding, fraction walls and number lines, with the presenter modelling the way language could aid and develop understanding of fractions. Teachers were asked to trial the fraction interview with their students. At the second session of the pilot program they were asked to complete a questionnaire which included the following questions:

- What were the benefits in using the fraction interview?
- What mathematical knowledge and understandings were revealed by students who did well on the fraction interview?
- What mathematical knowledge and understandings were revealed by students who had difficulties with the fraction interview?
- What general misconceptions or difficulties with fractions were highlighted by the interviews?

In response to the question, ‘What were the benefits in using the Fraction interview?’, teachers responded that using the fraction interview:

- revealed prior knowledge and students’ abilities;
- gave better insights into which tools and strategies students use; and
- identified the general misconceptions of students.

To the question: ‘What general misconceptions or difficulties were highlighted?’, the responses from teachers indicated that the fraction interviews highlighted:

- the confusion between whole numbers and fractions that students revealed;
- the difficulties with equivalence e.g. $\frac{3}{10}$ (the numerator and the denominator have to differ by 7 — further evidence of whole number thinking);
- that students could use a method or rule but had no understanding (Year 8);
- that students believed that the bigger the denominator meant a bigger fraction (Year 8);
- the lack of language to express fractions, e.g. ‘2 out of 3’ rather than two-thirds;
- that students saw fractions as the circular model rather than quantities; and
- that students would try to apply rules they had heard before, whether they were appropriate or not.

While teachers found the fraction interview very valuable in informing them about their students’ knowledge and misconceptions about fractions they identified the major difficulty as the amount of time taken to administer the one-to-one interview. In

response to their concerns a *fraction screening test* was designed with tasks that paralleled the fraction interview tasks wherever possible.

Fraction screening tests

A fraction screening test based on the more extensive Fraction Interview was designed for teachers in the SINE Years 5–8 pilot program. The purpose of the fraction screening test was as a broad assessment tool for teachers to use with a whole class group in order to identify areas of strengths and weaknesses and also to assist teachers to identify particular students who might later be interviewed one-to-one using the fraction interview. The fraction screening test was designed with tasks that paralleled the fraction interview tasks wherever possible. The tasks included contexts such as discrete items, lengths, fraction walls, and number lines. After the trial of the fraction screening test the authors decided to develop an easier version. This easier version was called *Fraction Screening Test A* (Pearn & Stephens, 2002a) with the original screening test called *Fraction Screening Test B* (Pearn & Stephens, 2002b). Fraction Screening Test A was designed mainly for students in Years 5–6 and for weaker students in Years 7 or 8. Fraction Screening Test B was intended for students in Years 7 and 8 or higher achievers in Years 5 and 6. There were twenty items altogether in each screening test, with eleven common items in each test and extension items in Test B only. In Table 2, the initial task from the fraction interview is compared to the items in Fraction Screening Tests A and B.

Table 2: Comparison of items from fraction interviews and screening tests.

Fraction interview	Fraction Screening Test A	Fraction Screening Test B
<p>Point to the picture of 4 lollies.</p> <p><i>Say:</i> This is one-third of the lollies I started with.</p> <p>How many lollies did I start with?</p> <p><i>Wait for student to respond.</i></p> <p><i>Say:</i> How did you decide?</p>	<p>This is one-half of the lollies I started with.</p>  <p>How many lollies did I start with?</p>	<p>This is $\frac{1}{3}$ of the lollies I started with.</p>  <p>How many lollies did I start with?</p>

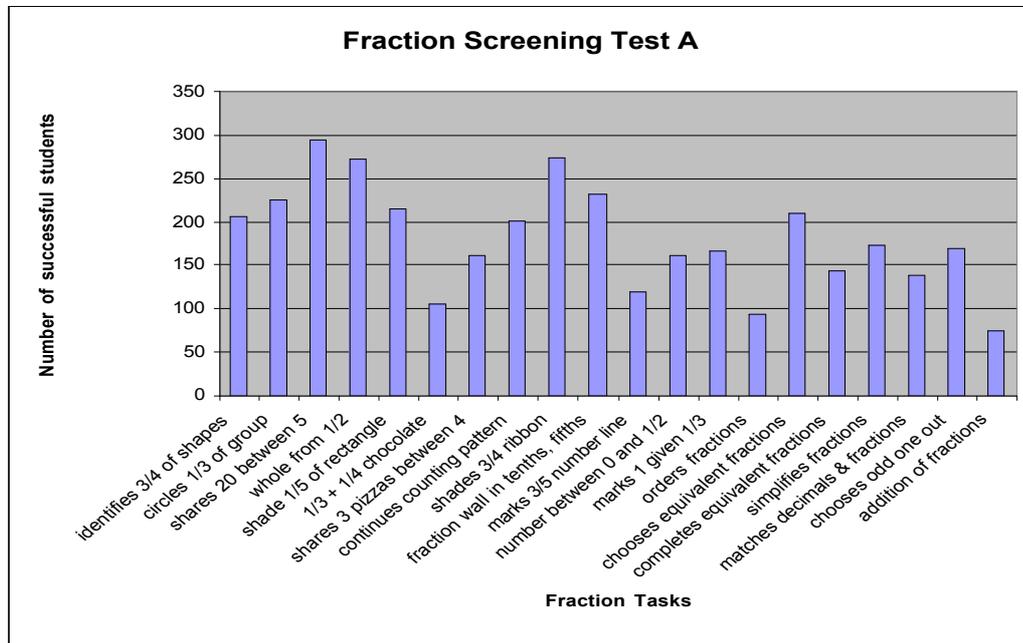
In the SINE Years 5–8 pilot program, primary and secondary teachers were asked to trial the fraction screening tests. Teachers sent in the results using a *class record* designed by the first author. These results were recorded by year level and school in an *Excel* spreadsheet.

Results

Teachers sent in the results for both Fraction Screening Tests A and B. In Table 3 we have recorded the initial results from Fraction Screening Test A from 8 schools, 17 classes and 307 students from Years 5–8. This table highlights both the strengths and

weaknesses of students who completed Screening Test A. Those items with less than 150 successful responses are discussed in Table 4 below.

Table 3. Results from the Fraction Screening Test A.



Students were successful with tasks presented in conventional contexts such as shading three-quarters of a ribbon and with the fraction one-half, for example, giving the whole from the half using discrete objects. They were less successful with tasks that involved fractions as numbers; for example

Put a cross (x) where you think the number $\frac{3}{5}$ would be on the number line.

Many students interpreted this question as requiring them to find three-fifths of the whole number line ignoring the numbers zero, one and two marked on the line.

Ordering fractions was a task which highlighted the common misconceptions of students. Examples of the difficulties and misconceptions when students ordered fractions from largest to smallest included students who believed that the larger the denominator, the larger the fraction, and ignored the numerators altogether.

For example, Student A ordered the fractions such:

$$\frac{1}{10} \quad \frac{3}{5} \quad \frac{1}{4} \quad \frac{2}{3} \quad \frac{1}{2}$$

Other students revealed that they believed that the smaller the denominator the larger the fraction. For example, Student B ordered the fractions as:

$$\frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{4} \quad \frac{3}{5} \quad \frac{1}{10}$$

Another group of students ordered primarily by numerators:

$$\frac{3}{5} \quad \frac{2}{3} \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{10}$$

There were many other variations to this task of ordering fractions.

While we have only given results for Fraction Screening Test A, the initial results for both fraction screening tests are very similar, as indicated in Table 4. Teachers are still sending in the data for the screening tests and these will be added to the database and further analysis performed.

Table 4: Difficulties highlighted by the fraction screening tests.

	Screening Test A:	Screening Test B:
<50% students successful	addition of fractions ordering fractions adding $\frac{1}{3} + \frac{1}{4}$ of chocolate bar marks $\frac{3}{5}$ of number line matches decimals & fractions	write sum for $\frac{1}{4}$ of $\frac{2}{3}$ addition of fractions (missing addend) shares 3 pizzas between 5 orders 5 fractions addition of fractions

Discussion

Teachers involved in the SINE Years 5–8 pilot program highlighted the difficulties and misconceptions revealed by both the fraction interview and the fraction screening tests. Listening to students explain their responses in the one-to-one interview revealed that students were appearing to understand fractions but were really getting correct responses due to flawed thinking. For example one student completing the counting pattern: $1\frac{3}{5} \quad 1\frac{1}{5} \quad \frac{4}{5} \quad \underline{\quad}$ gave the correct response. However when explaining her response she said, ‘It has to be two [fifths] because the first two are odd numbers so the next two must be even.’

Testing using either a one-to-one clinical interview such as the fraction interview, or paper and pencil tests such as the screening tests, highlighted the dependence of students of remembering rules and procedures. Sometimes this reliance was successful. For example when completing equivalent fractions they remembered that ‘what you do to the bottom you must do to the top’ and could successfully complete equivalent fractions. However when given the task of ordering fractions students were more likely to rely on remembering faulty rules and procedures.

In some cases students appeared to relate whole number rules to fraction concepts. For example, in the task which required them to match five fractions with their equivalent decimals some students matched 0.4 and $\frac{1}{4}$ because they both had a 4 in the number.

This was a common response for both the interview and screening tests from students at all levels from Years 5–8.

Secondary teachers commented that they had assumed that students understood basic fractions but the testing had shown that too many students were reliant on

remembering rules and procedures. Once teachers had used the clinical interviews and listened to the responses of their students they could make an educated guess as to the thinking behind the response on the paper-and-pencil test. For further confirmation they could ask the student after the test to explain their thinking as they completed the paper-and-pencil test.

Implications for teaching

Results from both the fraction interview and the fraction screening tests have revealed the reliance of students on rules and procedures to solve fraction tasks in various contexts. These results revealed the lack of comprehension or logical thought when fraction tasks were presented in contexts other than shapes to be shaded or traditional algorithms. These results have demonstrated the importance of developing the language of fractions as early as possible, and the need to develop a variety of models to demonstrate fractional thinking to ensure that students can partition, and develop a notion of fractions as numbers.

The fraction interview and Tests appear to show that too many children rely on rules and procedures to compensate for inadequate personal understanding of fractions. We believe that important fraction ideas need to be established formally much earlier in the primary school. Sharing situations in particular can assist with students' fraction understanding as well as their whole number thinking. Following Powell and Hunting (in press), we believe that the development of fraction ideas should start as early as possible in the primary school and not be delayed until the middle years.

This project is just beginning. More data is coming in from teachers from both primary and secondary Victorian Catholic schools. These data confirm the difficulties and misconceptions experienced by both teachers and students with fractions. Our next part of this pilot program will be to develop a resource to assist teachers teach the topic of fractions to students which will enable them to develop an understanding across a variety of contexts.

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Back and forth between classroom practice and mathematics education research⁺

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Mathematics is a very structured corpus of knowledge. To many students, mathematical knowledge appears as arbitrary. Often, students say that they do not know if their work is correct or not; they do not feel responsible for it.

Using their inner coherence, students can improve their personal abilities in mathematics in the context of the classroom. We give the students some tools to decide by themselves of what is wrong and what is correct. Then they may learn that mathematical truth does not depend on the goodwill of the teacher. In a rather unexpected way, the mathematics classroom becomes a place for freedom.

Introduction

I would like to talk to you about practice and theory. I am French and I will talk to you from the point of view of a mathematics teacher working in a French school. I would nevertheless be really surprised if you would not recognise yourself in most of the experiences I will share with you. The theme of this conference is ‘Mathematics ~ making waves’. We have waves in Nice, which is located on the seaside but these are not the ones I am expected to talk to you about.

For almost 37 years I have taught mathematics at all levels of high school. Twenty years ago, I got involved in mathematics education research, mainly to try to find answers to questions that came from my teaching. My research has always been based on the experiences I had in the classroom with students aged 15–18, and I tried, as far as it was possible, to bring back into the classroom the results of my research. My students have been used as guinea pigs for dozens of experiments. They never complained and I wish to thank them for their goodwill. My work is like the surf: going back and forth from classroom to research from research to classroom. What happened to me in this back and forth motion is that sometimes a big wave really swept away my beliefs on what students were, on what mathematics was, on what teaching was. These are the waves I would like to talk to you about.

⁺ Invited keynote address.

Something struck me during all these years devoted to teaching: if you listen very carefully to what students say, you find that they have good reasons to act as they do when doing mathematics, even if what they produce is not considered ‘correct’ by teachers. They have their *personal mathematical knowledge*. On the other hand, all mathematicians know that mathematics is socially constructed, that there is an agreement on what is correct and what is wrong in mathematics. This is a paradox: as teachers, how can we act so that personal knowledge coming from thirty different students can converge in order to become *mathematical knowledge* — shared, permanent, universal?

To deal with this paradox, my talk will be divided into three parts:

First we will look at the work of students and on this very common idea among mathematics teachers that: ‘Students are not coherent. They very often act without meaning.’ This applies in particular to algebra, which is seen as something which is easy to learn. We will look at the work of Leslie, a girl in Grade 10 and see how she deals with algebra, and try to examine her inner coherence.

Then, in the second part, we will turn to mathematics and try to find where its coherence comes from, what makes it universal. When we listen to students talking after a test they often say that they do not know whether their work was correct or not. The correctness of the solutions seems to depend on the goodwill of the teacher. The students seem to have nothing to do with it. Of course this does not apply to all students, some are quite good at doing mathematics; I am talking about a large number of them who perform rather badly. Is it possible to have those students control their work?

I will try to point out some characteristics of mathematics that mathematicians know about that helps them to control their work. We will observe that students generally know nothing about the structure of mathematics, that for them it is quite often a set of rules with no links between them. I will present experiments that my colleagues and I have conducted in some classrooms. Through these examples, we will see that we can teach in a way that the students become responsible for the correctness of their work. Then I will consider the role of the teacher, on the way s/he works in the mathematics classroom. I will show that it is possible to have students learn about all this knowledge that mathematicians have and use to produce mathematics.

In the third part, which will be the conclusion, we will see that when students are in a situation to distinguish, by themselves, what is true and what is false when they do mathematics, things are no longer arbitrary. The authority is no longer in the hands of the teacher; it comes from the mathematics itself. The mathematics classroom is then a place for freedom.

The inner coherence of students

Some examples

As teachers we very often think that students are not coherent when they do mathematics. We have plenty of examples of this fact; they write:

$$x + 2 = 3 \text{ gives } x = 3 - 2$$

but also:

$$2x = 3 \text{ gives } x = 3/-2.$$

How many teachers can testify of such calculations done by students which appear as if they were just writing things at random.

If we listen to what students have to say about their knowledge, we are quite surprised by the way it is structured. We will describe the means by which we collect this sort of information later, and we will consider a statement made by Leslie (aged 16):

The square of any number is always positive; a positive number has a plus sign before it. If we look at the expression $(a - b)^2 = a^2 - 2ab + b^2$ one can observe that b^2 has a plus sign before it and thus is positive as we know.

The theory

Algebra often appears to students as a set of rules with no links, no necessity, which could be changed if one wanted to. It seems impossible to memorise all of them. At least in France, it is quite common to hear teachers say: 'There is nothing to understand in algebra, one simply has to learn the rules'. If this is the belief of the teacher, the students will certainly think the same. Therefore we must not wonder about their knowledge seeming to have no coherence. The problem is not the lack of coherence: the problem is a different coherence, each student taking care of it in a different way. We have developed the notion of 'Local Bits of Knowledge' (LBK) which help us to understand this phenomenon.

Let us look at some examples of LBK.

Local bits of knowledge

In one of our studies we observed that students use two rules to compare decimal numbers:

R1: 12.18 > 12.6 because 18 > 6.

R2: 8.898 < 8.78 because it has more digits after the point.

Several things are interesting about these rules: quite often they give the correct answer (for *R1* if the number of digits after the point is the same). When they are contradictory, one of them gives the correct answer. When they give the correct answer it is possible to make a correct mathematical demonstration of it. Another interesting point is the comparison between the frequency of these rules in different countries. In France, decimal numbers are taught using units (meters, centimetres). *R1* is very frequent and is correct within the set of decimals having the same length. In the US, students learn first about fractions then about decimal numbers. For fractions, $8/898 < 8/78$, the longer the denominator the smaller the fraction. *R2* is the same rule applied in an incorrect way to decimal numbers.

Algebra is an inexhaustible mine of such LBK, and it is quite easy to understand why, as we will see in a moment. Here are two very simple examples:

if $a < b$ then $ax < bx$.

$x^2 > x$.

The three characteristics of LBK

There are three characteristics of these LBK, which justify the name we gave them. There is a 'domain' in mathematics in which they are valid: that means that, inside this domain, first they are true from the mathematical point of view; second, if used, they give the correct result; third they are coherent for the students. If used outside this domain they are not valid; they do not give the correct result. That is why we speak of knowledge and use the word local. Of course the students ignore the limits of the domain. When they learn something new, they learn it with the knowledge they already possess. They use the new knowledge as they would use the previous one. That is the reason why they make errors. The knowledge ' $x^2 > x$ ' is true for positive integers. That is certainly where it comes from. If one uses this knowledge only for these numbers the results of the computation will be correct. Many studies have shown that students aged 14 or 15 seem to ignore that other numbers exist. These studies also highlight the fact that teachers use mainly whole numbers (positive or negative) in examples; for students this knowledge is coherent with their knowledge on the numbers they are familiar with. Concerning the other LBK 'if $a < b$ then $ax < bx$ ', in the French curriculum, the rules for inequalities are studied one year for positive numbers and the year after for negative ones. No wonder if students memorise this local rule!

The means to work on these LBK

My purpose here is to show that it is possible to make students work in a way where it is not the teacher who tells what is right and what is wrong, but the students themselves.

We have been using interviews starting with a paradoxical injunction. What is a paradoxical injunction when one teaches mathematics? We are facing students who have difficulties in mathematics; they complain that they are always getting it wrong. Generally the teacher asks them to 'be careful, think before writing, remember the rules you are supposed to use' in order to have them write computations that are correct. We are asking them to write something false. We propose an expression and ask them to write something that is false — that is *always* false. 'What! a teacher who wants me to write something false? It should be easy for me, I always do it.' The fact is that the students find it difficult to do so. They do write many false things but, when writing them, they think they are doing well, or at least they are generally trying very hard to compute correctly. Faced with a request from the teacher to deliberately write a false expression, they are helpless. They do not know how to do it. Nobody ever taught them to write false expressions. They have to use all the mathematics they know — their LBK — first, in order to write something false, then to explain why it is false, and then to find out whether it is always false.

Here is an extract of the beginning of such an interview. Leslie is a 16 year-old girl; the interview took place at the end of the school year and she has had average results in

mathematics. We gave her an algebraic expression and asked her to write something false:

- Teacher: I write ' $7x/7+x=...$ ' and you write something to make it false.
- Leslie: $7x/7+x = 7x/7x$, and this is false.
- Teacher: How do you know that it is false?
- Leslie: Because I changed a + sign into a \times sign and I know that multiplication is different from addition; moreover, if I try with $x=1$, it makes $7/8$ and this is not equal to 1.
- Teacher: Is it always false?
- Leslie: You should have $7x=7+x$ and you know that this is false... Logically, if we go on, we should be able to find x .

As one can observe, Leslie had chosen the ground on which she was going to work: difference between addition and multiplication and equations ('Logically, if we go on, we should be able to find x ').

At this time, she found herself in a contradictory situation. There was a conflict between two local bits of knowledge:

- LBK 1: If one changes a computing sign in an algebraic expression, then the expression changes, which means that its numerical value changes according to numerical values of x . (The formal aspect of the expression changes and so does the expression.)
- LBK 2: If there is a sign '=' between two algebraic expressions using the letter x , this is an equation and some computations should 'logically' give a value for ' x '.

The first of these two local bits of knowledge is an answer to the paradoxical injunction 'write something false'. The student knows a rule; to produce something false she modifies the rule but the change is only formal. Yet she knows no 'rule' that will produce something that is always false.

The student generally forgets, very quickly, about the injunction and starts working on the problem s/he has posed to her/himself. This problem is not expressed in a way familiar to the student; it is not 'solve the equation...', which Leslie could do very easily, but it is her problem. She is in charge of it. Of course, she could say, 'I'm not interested in mathematics; I don't want to find out anything correct or not.' We take care, before starting the interview, to make sure that the student is willing to work with us, so if s/he is not interested s/he does not start. Once the interview is started, there are no problems.

As the interview went on, Leslie kept talking about equations and about finding x , but she could not do it. What was interesting for us was that she was no longer able to solve an equation as simple as $7x = 7 + x$; she did not seem to have adequate knowledge any more. She tried several computations to solve this equation but she always failed and she kept repeating that $7x$ is different from $7 + x$. At the end of the interview she seemed to have convinced herself that $7x/(7+x)$ is always different from $7x/7x$.

Let us look precisely at what happened during this first interview. Trying to answer to the paradoxical injunction, the student takes us directly to some point that is difficult for her: something (the equation) that she can deal with only if it is formulated in the exact way she is used to. The student shows us that the formal aspect of the expression is what is important. She does not seem to know that algebraic expressions stand for numbers, that the same number could be written in different ways. Neither does she know that two expressions could be equal or different depending on the value of x . Such knowledge is important to work in algebra, and many students know nothing about it. We will come back to it in the next part of this presentation. Something else happened during this interview: Leslie was struggling with mathematics that was meaningful for her. She was eager to solve the problem of the equality being correct or not, about finding x . As we will see in the second interview, she knew perfectly well what she was doing.

Second interview

The second interview took place twelve days later and Leslie recalled easily what the problem was:

Leslie: I had to know if $(7+x)/7x$ could really be equal to 1.

After some time, when she had been working on the question of addition and multiplication, Leslie could formulate very clearly a new knowledge about equations:

Leslie: Well, we have an equality, and we don't know whether it is false or not... and so we shall try to find the x in $(3+x)/4x$, so that if we place it into $3x/4x$ we can see whether it is equal or not.

This local bit of knowledge has nothing to do with the very formal one that we saw at the beginning:

LBK 2: 'an equation is made of two algebraic expressions linked by an '=' sign'.

Leslie solved the equation $(3+x)/4x = 3x/4x$ — not very quickly, but she did solve it; then with $(7+x)/7x = 1$ there was no problem.

With the second interview we can see that when she was confronted with a specific situation, Leslie quickly recalled the tools she possessed to solve a problem. When she had solved one equation, she had no difficulty in solving another one.

It was possible to make her elaborate on her own knowledge by herself: she constructed her personal way to talk about an equation. Confronted with the reality 'multiplication and addition can yield the same result', which she did not believe at first, she had to construct a system which could take this into account.

As we can see, it is possible to help students modify their LBK. Doing this, Leslie was not incoherent; we discovered how her knowledge was structured, and the reasons why she experienced difficulties. What is more important for us, is that the student has the ability to decide, by her/himself, what is correct and what is wrong in mathematics. The teacher leads the student in this work, but never says, 'This is correct; this is not.'

What knowledge is needed to act as a mathematician?

A melting pot of personal opinions

Mathematics is socially constructed, yet every subject has to construct its own mathematics. Truth in mathematics is the same for everyone, everywhere, and will always be. Leaving aside some moments when philosophers (Russell, Gödel) have questioned mathematics, and we will not talk either about non-Euclidean geometries; some axioms may be changed but once the axioms are chosen and fixed, mathematics is the same for everyone.

We have seen that the inner coherence of students can lead to very strange LBK. How will it be possible to make students agree on the same knowledge?

This is a paradox: knowing that their knowledge is constituted by LBK, we want to give each student the opportunity to decide by themselves of what is right and what is wrong. At the same time we, as teachers, want to build knowledge which is shared not only by the students but also by the mathematicians.

How to shift from a collection of personal opinions to a truth accepted by everyone? We have seen, in some experiments, groups of students who agree on false conclusions. Would not the easiest solution be that the teacher says what is wrong and what is right?

The mathematical 'reality'

Many people see mathematics as a place where many things are forbidden (it is forbidden to divide by zero) or, conversely, compulsory with strict rules — for multiplying relative numbers for instance. The point is not so much a problem of forbidding but a matter of coherence. One cannot walk through a wall without breaking it, so human beings do not try to do it. This is an example of how physical reality resists.

We say that mathematics 'resists' in the same way.

A mathematical result, within a theory, cannot be changed: it has to be what it is. In a set of real numbers, although it would help many students, we cannot decide that $(a+b)^2 = a^2 + b^2$. If we did, then the whole mathematical knowledge would become contradictory and it would be very simple to prove that $1 = 2$, and that all numbers are equal. So there exists something that is imperative in mathematical knowledge, that cannot be changed. Mathematical results are necessary. This is what we refer to as the mathematical reality. Many students ignore this and are not at all surprised if two different people find different results when solving an equation. Some students told us, 'Well, you find $x = -1$ and I find $x = 2$; of course it's different, but we did not use the same rules!'

What makes mathematics 'resist'? The way it is structured, with the rules of the mathematical game. There are plenty of these that are mostly consequences of rules in logic. We will look at some of them.

One of the easiest to state, is that one thing and its opposite cannot be true at the same time. Others have to do with what is called denotation and we have met them when

looking at Leslie's work: Leslie knew that when you have a letter in an expression, it has the same value all over the expression. Not all students know that. We find students who believe that the number represented by x in the equation $2x + 7 = 3x + 3$ could be 1 in the left part and 2 in the right part of the equality. When she started working, Leslie did not know other aspects of this knowledge. As we said before, she had difficulties in dealing with the fact that the same number could be written in different way ($7x$ and $7+x$). She did not know either that two expressions could be equal or different depending on the value of x . These are some examples of the knowledge that is necessary to work in algebra and that is not, in general, written in the books, nor clearly pointed out by the teachers.

Now we have to distinguish between the computing rules in algebra and this other knowledge that is necessary to learn mathematics. Learning mathematics is not just learning a set of computing rules. One also has to learn how to do mathematics, how to play the mathematical game that is a social game with rules accepted by all the mathematicians. The question is not so much, 'Are we allowed to do this?', as many students ask; the question is rather, 'When doing this, do I respect the coherence of mathematics?'. Another way of saying this is: 'When I use $(a+b)^2 = a^2+2ab+b^2$ is it just one of my teacher's fads, or is there some other reason for it?'

How can we make students play the mathematical game in the classroom? For clarity, we will give different names to the rules of algebra (the computing rules) and to the 'rules of the mathematical games'. The rules of algebra are theorems, from a mathematical point of view, so we will call them theorems and keep the word 'rules' for the rules of the mathematical game — those that make mathematics resist.

We believe that learning to play the mathematical game has another major consequence: to understand and memorise the theorems, it helps a lot to know the rules of the game. It is much easier to memorise results that you know have a coherence, that you know how to control, rather than memorising some sort of arbitrary catalogue of results.

We will now give two examples of activities we use in the classroom to try to make students work on these rules of the game.

Two examples

The diagonals in a cube (age 14–15)

Several teachers experimented with the lessons I am going to talk about in Grades 9 and 10. In French mathematics education, this type of working is known as 'scientific debate' and its purpose is to make classes work in a way (relatively speaking) that a group of mathematicians would work. In this situation the role of the teacher is to organise the debate, not to give answers or hints about the problem being considered, except at the end of the lesson as we will see later.

This example deals with the diagonals of a cube.

The teacher starts by writing a conjecture on the board: 'The diagonals of a cube are perpendicular'. Nothing else is written, no picture is drawn. Some minutes (about ten)

are devoted to some personal work, during which students are supposed to decide by themselves, using whatever method they wish, to decide whether the written statement is true or not. Students then have to declare whether they believe the statement to be true, false, or if they do not know. The teacher asks one of the students who said 'I don't know' the reason of this answer: 'Can you explain why you could not decide? Is there anything you wish to know that will help you make a decision?'.

Then the debate starts, the students wanting to explain their solution and to convince the others. The teacher stays in the background, sometimes summarising the things that have been said so that nothing important is forgotten, and making sure that nobody who is experiencing difficulties with the concepts is left out. The teacher might say: 'X said that we should consider... and we did not do it; could we go back to this...'. In reporting the debates, I will focus on this knowledge that is not 'only mathematical', that is concerned by the rules of the game.

- Quite a few students, in all classes, did not really know what the diagonals of a cube were, confusing them with the diagonals of the square sides of the cube. They had to work to produce a clear and shared definition of the diagonals of the cube. (One can imagine what would happen to a child who had this confusion if the teacher demonstrated the property).
- It occurred to them that a drawing, with the points named, could help.
- A student was sure that the diagonals were perpendicular because they appeared so when one looked at the cube from above.
- Some students thought that it might depend on the cube. They believed that all cubes were different, because they were drawn by different students, and that the property could depend on the 'shape' or the 'size' of the cube. One student explained that they were working on a 'mathematical' cube that was an abstract one; the ones drawn or built by the students might not have the properties of the mathematical cube because they were not 'ideal'.
- One student built a very neat cube made of paper and managed to insert some metal rods to represent the diagonals; she then passed her cube around the classroom and students looked at it. She was convinced that the diagonals were perpendicular because she 'saw' them that way in her model.
- After a while, almost all students agreed that the conjecture was incorrect; the ones who had been the most in favour of it asked strongly for a demonstration before changing their minds: 'OK, we agree, but you have to demonstrate it'.

At the end of the lesson, the teacher again took control, summing up what the students had to remember:

- the mathematical result (of course);
- precise definitions are a necessity to do mathematics;
- mathematics deals with abstract objects, which are representations of physical objects;
- observation is not enough and can lead to errors;

- to convince others and to be sure of a result, a demonstration can be needed.

Inequalities (age 16)

The purpose of the lesson I am going to describe was to make students both reflect on an error that is frequent in algebra:

If $a < b$ then $ax < bx$.

and learn about the necessity of theorems.

I will describe the method I used and give an analysis of it. Then I will show and interpret the responses of some students.

The problem was the following:

Solve the inequality $3/x > x+2$.

All methods are valid.

Describe the successive steps of your work.

You must be as sure as possible of the result you give.

Think of a way to convince others that your result is correct.

There is nothing new that the students need to learn. They are supposed to know everything that is necessary to solve the problem. We are trying to correct some very common and persistent errors.

One idea is quite simple: this error would lead the students to a conflict that they would solve and they would memorise the correct knowledge. The idea of creating a conflict between students is not new. Most of the time the students manage to come to an agreement, which is either correct or not from a mathematical point of view. Then the teacher validates the correct answer and the students learn the correct result. We have all tried it and we know that it does not work that well.

We tried something that at first glance does not appear to be different. We keep in mind three ideas I already mentioned:

1. The students' knowledge has an inner coherence.
2. It is possible to make the students feel responsible for the correctness of the mathematics they perform and this leads them to confront themselves with the mathematical reality.
3. Learning mathematics is learning, simultaneously, theorems and rules of the mathematical game.

I want to stress the fact that this lesson was not unique in the work of the class. We have had several opportunities to perform scientific debate or similar activities. Every time I propose, in the classroom, some work of this type, I insist on the fact that although it does not look like a maths class, the students will learn mathematics. They trust me.

The work was organised according to the following schedule:

- Session one and session two: 1 hour 30 each with each half of the class. 30 minutes of personal work (phase A) and 1 hour of work in small groups of 4 students (phase B).
- Session three: 1 hour 30 with the whole class for a synthesis (phase C).

Phase A

The work in small groups is prepared for by some time devoted to personal work. This time is important: while they were working alone the students had the possibility to use incorrect knowledge and to make an expected error. We expected some of them to use a graphical method (for instance using a calculator, but this was not necessary as they could draw the curves by hand) and find the correct result if they managed to read the solution correctly on their calculator. There were some opportunities to make mistakes: the fact is that many students did make errors but, at the same time, they were convinced that they were doing well.

During that time the teacher looked at their work and organised the small groups in order to generate conflicting ideas inside them.

Phase B

When the small groups met, the students were really eager to defend their opinion and to convince others of their argument. During that time the students had to:

- determine what the correct solution was;
- find out where the errors came from;
- experience the necessity of the correct rule (meet the mathematical reality);
- come to an agreement on a shared knowledge, as any of them could be asked by the teacher, to explain the work of the group to the whole class.

This last point means that the students had to shift from a personal opinion to a scientific discourse. The situation and the contract of the work in the classroom obliges them to decide between two contradictory results: they cannot go on saying: 'You find this and I find something else, but it doesn't matter because we didn't use the same method of solving'. They have to make a decision to demonstrate their solution to the other groups and try to produce something that is correct, abandoning what is not.

Phase C

When the whole class met, some students came to the blackboard to report on their group's work. The class had a discussion on the different ways of solving the problem. Then the teacher summed up the mathematical results and some rules that are necessary to work in mathematics:

- To solve an inequality one can either use an algebraic method or a graphical method, but the result must be the same.

- To determine whether the solution is correct one can use values of x ; this can be used to show that the result is not correct, but cannot be used to demonstrate that it is (counter-example).
- The rule ‘if $x > 0$, if $a < b$ then $ax < bx$; if $x < 0$, if $a < b$ then $ax > bx$ ’ is necessary (cannot be changed).

I will now describe the work of some of the students to show how they dealt with the contradiction they faced. Doing this we will see that most of them, when in charge of a problem, are quite able to imagine ways to get the correct answer. However they do not always produce correct mathematics. We’ll also see that not all students accept being responsible for the correctness of the mathematics they are doing.

A. Mathieu and Julien

They used values of x to check the correctness of the solution and determine the right answer.

For a solution using graphs, three steps :

1. get the curves.
2. read on the graph.
3. ‘check with computation’

I use values of x in $]-\infty, -3[\cup]0, 1[$

ex : $x = -4$

$$3/x = 3/-4 = -0,75 \quad x + 2 = -4 + 2 = -2$$

I see that $3/x > x + 2$

I think my result is correct because when I check I see that the values give the good result.

Here are the curves they used:

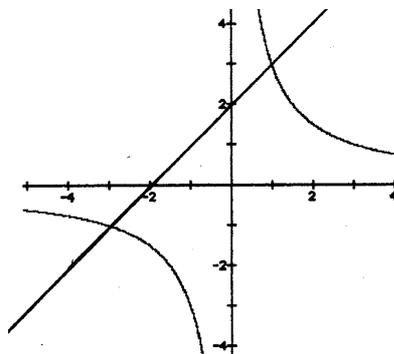


Fig. 1. $y=3/x$ and $y=x+2$

B. Magali, Mélanie and Renan

Mélanie had first solved graphically $3/x - x - 2 < 0$. Then she multiplied by x and solved graphically $x^2 + 2x - 3 < 0$. She had two different results. Through the discussion they found that they had to change the sign in the inequality if x was negative. They were using one of the two parabolas $y = x^2 + 2x - 3$ or $y = -x^2 - 2x + 3$ and solving, graphically, $y < 0$. It took them a rather long time to understand that they had to use one half of each parabola depending on the sign of x . They said ‘every time there is a parabola, it does not work’ which meant that the hyperbola gave the correct result and the parabola did not. This is not surprising for us but for them it was.

We first compared our results: Renan and I had been doing the same thing and we had the same result: $]-\infty ; -3[\cup]0 ; 1[$. Mélanie had a different result.

(She had first solved graphically $3/x - x - 2 < 0$. Then she multiplied by x and solved graphically $x^2 + 2x - 3 < 0$.)

In fact she has two results and her two results are different.

We have to understand what’s happening.

In Mélanie’s second solution we tried to change the sign of the inequality, because we don’t know the sign of x . If x is negative we have to change the sign, but still we don’t find the result Renan and I found. We are almost sure this is the correct result, but we can’t get it when we have a parabola. Every time there is a parabola, it doesn’t work.

The graph gives the solution $]-\infty ; -3[\cup]0 ; 1[$.

Eventually we managed to find the same result using the parabolas but it was really difficult.

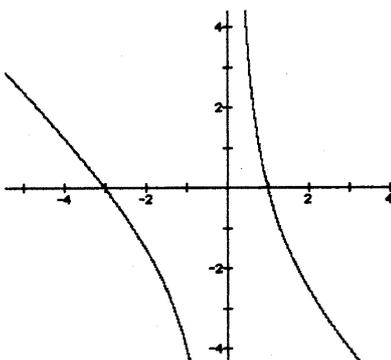


Fig. 2. $y = 3/x - x - 2$

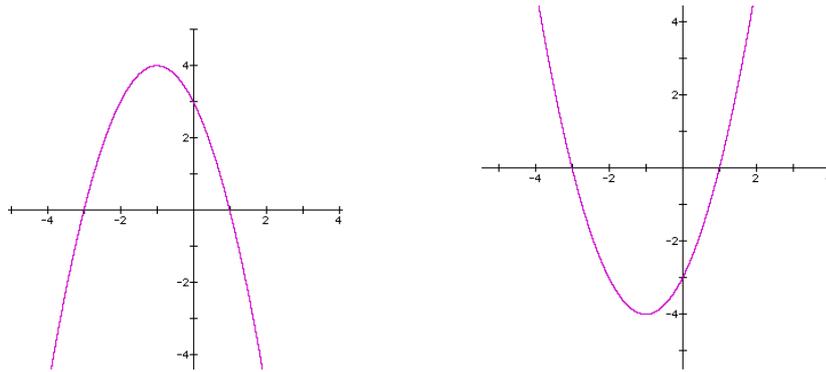


Fig. 3. The two parabolas

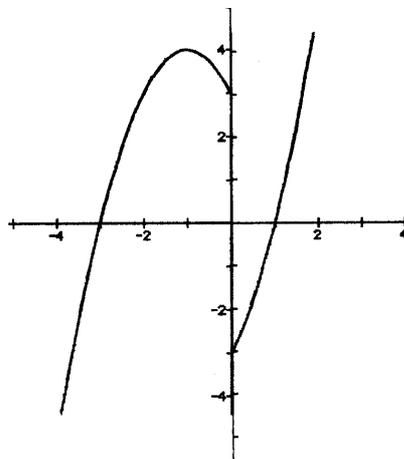


Fig. 4. The two half parabolas

C. Julien and Jacques

Jacques managed to find the correct result but he was still doing several mistakes such as writing $-x$ to show that x is negative! The following shows that Julien deals more correctly with the same difficulty.

Julien and Jacques had changed the inequality to: $x^2 + 2x - 3 < 0$ and got the result using a table:

Table 1

x	$-\infty$		-3		1		$+\infty$
$x - 1$		$-$		$-$	0	$+$	
$x + 3$		$-$	0	$+$		$+$	
$f(x)$		$+$	0	$-$	0	$+$	

So their solution was $[-3 ; 1]$ which of course was different from the result obtained by those who had used the graphs: $y = 3/x$ and $y = x + 2$.

Julien wrote:

If x is negative one must change the sign in the inequality, and we then have inequalities to solve:

$$\frac{3}{x} > x+2, \text{ that is } 3 - x^2 - 2x \geq 0$$

$$\text{and } \frac{3}{x} < x+2 \text{ that is } 3 - x^2 - 2x \leq 0.$$

(One can observe that he does not change the sign when multiplying by x).

He solves them, putting a zero in the table and gets the solution for $x < 0$: $]-\infty; -3[$.

Table 2

x	$-\infty$	-3	0	1	$+\infty$	
$f(x)$	$+$	0	$-$	$-$	0	$+$

Jacques solves in a different way, starting from:

Table 3

x	$-\infty$	-3	1	$+\infty$		
$x - 1$		$-$	$-$	0	$+$	
$x + 3$		$-$	0	$+$	$+$	
$f(x)$		$+$	0	$-$	0	$+$

He says:

When x is negative, one must change the signs, because $-x$ is $(-1)x$ and, so, there one negative factor more and the table becomes:

Table 4

x	$-\infty$	-3	0	1	$+\infty$		
$x - 1$		$-$	$-$	$-$	0	$+$	
$x + 3$		$-$	0	$+$		$+$	
-1		$-$	$-$				
$f(x)$		$-$	0	$+$	$-$	0	$+$

and the solution of the inequality is: $]-\infty; -3[\cup]0, 1[$.

It is clear than Jacques manages to find the correct result but that he is still doing several mistakes such as writing $-x$ if x is negative!

D. Nathalie, Florent and Emmanuel

The work of these three students is very interesting because they are all three good students and Nathalie was the only one in the class who thought, at the very beginning, of distinguishing between $x < 0$ and $x > 0$. But she made an error in the computation and did not get the correct result. Emmanuel made the expected error and Florent used a graphical method and got the correct answer.

Then they stopped working. They had found one error in Nathalie's work and they did not see any reason to go further and see if they could find the correct result algebraically. They were waiting for the teacher to confirm the solution and to show the correct algebraic solution. They did not believe that their work could be to go further and be sure that they could do it by themselves. They did not feel responsible for it. Their idea is that their job, in the math classroom, was to solve. If it was incorrect it was just too bad. For all of them, most of the time it was correct and that was enough.

A short summary of these experiments

The students in the classroom are asked to perform some sort of work that is unusual. The teacher advises them that although it is unusual, they will learn mathematics. The students trust the teacher on this point.

There are three levers one can act on: the inner coherence of students, the confrontation with the mathematical reality (through the responsibility given to the students of the correctness of the solution), the simultaneous learning of theorems and 'rules'.

The device permits students to have personal opinions due to the time for personal work. The work in groups which follows transforms opinions into a scientific discourse directed to the whole class. The teacher can then summarise both the theorems and the rules.

Some students manage to get a correct answer without really learning the expected knowledge, but the teacher can act on that. It is more difficult to try to get the students who will not take the responsibility of the correctness of the solution to change their beliefs on what a mathematics class is.

The role of the teacher

I will take some time to examine the role of the teacher in the different activities I presented. There are many similarities, although in one of them the student was alone with the teacher and the others were performed in a classroom context.

The teacher never gives her⁵ opinion. In the interviews, as well as in the classroom, the teacher, as a mathematician, stays in the background — but this is *as a mathematician* only, and it is only while the students are working.

If we go deeper into this analysis, we can, very roughly, identify three periods in these teaching episodes.

⁵ Although different people, most teachers referred to in this paper were women.

Before the teaching, the teacher has to decide which work she will ask the students to perform. We, as teachers, know that not all problems are suitable for specific work. We have to choose them from a mathematical point of view and also from a pedagogical point of view. If one wants to work in a particular way of working with students, the choice of the problem has a very great importance. Generally speaking, to produce the expected results, one should choose a problem which is likely to be solved using some typical and very resistant LBK. The students are rather sure of what they 'know'. Then they will want to persuade the others that their result is correct. As we have seen, the confrontation between different students is an important means to have students learn mathematics, as long as students first have had time to make up their mind on what their own opinion was.

During the second period, the teacher is less active from a mathematical point of view: her job is then to organise the debate if we are in the classroom situation, or to have the student experiment on her/his mathematics if we are in an interview. This means that the teacher has to make very quick decisions about what should be emphasised and what should be left aside. In neither situation does the teacher say, 'This is correct, this is not; this is a good idea, this is not, it won't take us anywhere'. At the same time, it is obvious that the classroom discussion cannot be completely free. Some threads have to be followed in such a way that the students do not lose track of what is going on or jump from one idea to another one without exploring them completely. The teacher keeps record of what has been said, writing on the board for instance. She can also come back to some previous statement made by a student but she does not give her opinion. If the discussion is similar to a tree with branches going in all directions, the teacher has to cut some branches in order to have the other ones develop to reach the correct result.

When this part of the lesson is finished, the teacher again takes the role of teacher and mathematician. She speaks on behalf of all mathematicians, from yesterday, today and in the future. She says that the solution that the students agreed on is the correct solution. It is not only an agreement inside the group, it would be valid in all groups of mathematicians because they came to this solution using the rules of the mathematical game, and in doing this they have been doing mathematics.

Conclusion: On the way to freedom

Now we could ask the question: why use such a complicated scenario, if at the end the teacher says, 'Well, this is correct'? Should not the teacher, from the beginning, guide the students on the correct path?

When we perform this sort of interview or lesson, we give the students the opportunity to make their own choices, to try what they wish to try and confront it with other students' knowledge, or to what they know from another point of view. We saw that Leslie came up against a contradiction. For some time this impeded her from doing things that she generally could do quite easily. At the same time she had to reflect on *her* understanding of what an equation was. She chose (although she did not know she was choosing) the mathematics she wanted to work on. This she did by performing

some actions she was sure she understood (trying with different values of x , $7/8 \neq 1$). Thus she managed to decide that some things were correct and others were not.

The situation when working in the classroom is somewhat different. The main point is again to give the students the opportunity to decide, by themselves, what is correct and what is not. This is easier in the classroom. The confrontation between students who have their personal opinions about the problem and desire to defend it, leads them to find, under the rules of the game, the means to overcome the contradiction between themselves. An inequality cannot have two different sets of solutions; if we try a number from the set of solutions, it has to satisfy the inequality, etc. The consequence of this is important: the correctness of the solution does not depend on the teacher. It depends on the mathematics. Where does the authority come from? It comes from the mathematics itself. All students can say, 'This is wrong, this is right,' — even the poorest ones. Any student can have the opportunity to say once in a while, as Leslie said, 'This is wrong, because if I try with $x=1$ it gives $7/8$ and I know $7/8$ is not equal to 1.'

The rules of the mathematical game give everyone freedom to know whether or not what they are doing is correct. Of course this needs special organisation in the classroom, a well-chosen problem, a teacher who can perform such a lesson, and time. We, as teachers do not have to do this for every lesson, just from time to time, so that the students are in a situation to experience it. If they know that they have the possibility to decide about the correctness of their mathematics, they are more likely to accept that most of the time the teacher is in charge of it. Things are no longer arbitrary. The students are free.

The very strong structure of mathematical knowledge is a guarantee of freedom for all of us.

Freedom is understanding necessity (Hegel).

Learning to teach 'otherwise': Pedagogy as conversation*

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As educators, when we have a vision of practice that involves teaching differently, we need to conceptualise it to make it more accessible for teachers' professional learning. Learning to teach 'otherwise' refers to a way of thinking about teaching and learning mathematics that involves the use of different forms of conversation to create appropriate pedagogical relationships with students. This paper uses the literature on productive pedagogies to explore what it would take to teach mathematics 'otherwise'. It also provides a framework to illustrate what pedagogy as conversation would look like, sound like and feel like in mathematics classrooms.

In recent times, the term 'reform' has been an integral part of the discourse related to mathematics education. An underlying theme of many reform movements has called for a paradigm shift towards teaching and learning for understanding. Such visions of practice view mathematics as a way of ordering and explaining our world in relevant and meaningful ways (Carpenter & Lehrer, 1999; Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier & Human, 1997). To some extent, teaching and learning for understanding involves imagining mathematical relationships as if they could be 'otherwise'. One way of characterising teaching 'otherwise' is to imagine a shift in teaching from telling to listening. This paper describes a model for conceptualising the pedagogy of mathematics that illustrates how a shift from telling to listening might be enacted.

Pedagogy as conversation: Listening more and telling less

For teachers, a transformation in thinking about mathematics in different ways requires a change in how we view our learning relationships in classrooms. A focus on *listening more and telling less* calls for a more dialogic and conversational approach towards the pedagogy of mathematics. I have used the term *pedagogy as conversation* (Smith & Lowrie, 2002) to conceptualise and explicate such an approach. In doing this, I hope to open the door wider on the professional conversations related to the pedagogy

* This paper has been subject to peer review.

of mathematics. This would create a step towards developing a metalanguage for talking about the teaching and learning of mathematics, similar to that called for in the research related to productive pedagogies (Education Queensland, 2002; Lingard, Mills & Hayes, 2000).

Productive pedagogies: A mathematical focus

The Queensland School Reform Longitudinal Study (Education Queensland, 2002) documented specific pedagogies that appeared to make a difference to both academic and social outcomes for students. In their comprehensive study they named a model of pedagogy that consisted of four dimensions: intellectual quality (enhancing higher-order thinking and deep understanding); connectedness to the world (emphasising a problem-based, contextual focus); supportive classroom environment (enhancing a student-centred, engaging and positive environment); and the recognition of difference (acknowledging cultural diversity and the need for inclusive practices that lead to a sense of community and identity). While it is acknowledged that few of the classrooms observed in the Queensland study related to the pedagogy of mathematics, this paper seeks to connect some of the emerging metalanguage related to productive pedagogies with the findings from a previous study (Smith, 2000) to highlight a shared focus on sustained pedagogical conversations that can have a mathematical focus.

The term *pedagogy as conversation* emerged from my reflexive inquiry into the process of learning to teach mathematics that I have undertaken with the prospective primary teachers I worked with. Perhaps the most crucial aspect of this characterisation is the term ‘pedagogy’ and what it represents. In more recent times, the term pedagogy has referred to appropriate teaching and learning relationships with students (van Manen, 1999), or simply ‘classroom practice’ (Lingard et al., 2000). I argue that the aspect of *assessing* learning plays an integral role within a pedagogical relationship and should not be left out of any discourse related to productive pedagogies. The notable absence of any explicit reference to assessment within the four productive pedagogies suggests that assessment still remains the poor relation that is invited into pedagogical conversations as an afterthought. If teaching occurs only to the extent that learning occurs, then surely we need assessment to play a vital role in our pedagogical relationships with students.

Interestingly, some implicit support for the inclusion of assessment when referring to the term pedagogy comes from the Queensland study reported in Lingard et al., (2000). In their paper, the authors referred to the need for ‘professional conversations about how assessment can complement productive pedagogies, not work against them’ (p. 109). I would go one step further and argue that assessment must be an integral and explicit part of any productive pedagogies that describe relationships and ways of being in classrooms with students.

In the literature related to mathematics education, more explicit support for the inclusion of assessment in any conceptualisation of pedagogy comes from work done by Kerr Stenmark (1989) on assessment alternatives that promote learning. Similarly, Bright & Joyner (1998) referred to the phenomenon of ‘classroom assessment’, which is concerned with the type of assessment whose main purpose is to inform instruction

and to communicate what students know, understand, and can do in mathematics. This phenomenon is similar to what Clarke (1997) describes as ‘constructive assessment’. A major goal of classroom or constructive assessment is to understand learning from each student’s point of view on a day-to-day basis so that teachers can improve instructional practices.

Perhaps the strongest support comes from Morgan (2000) as she reviewed mathematics assessment research from a social perspective. She stated that, ‘there has been increasing recognition of the role that assessment structures play in influencing what happens in classrooms, either hampering or ‘leading’ efforts at curriculum reform’ (p. 227). Morgan (2000) urges us to view assessment as an ‘interpersonal interpretative practice, not a scientific instrument’ (p. 237). She goes on to suggest that understanding the ways in which assessment works in mathematics classrooms will depend on the complex relationships between individual teachers and students. Her social perspective can also contribute to recognising and challenging the discriminatory effects of assessment practices.

Clearly, the three educative processes of teaching, learning and assessing exist in a reciprocal pedagogical relationship. Assessment practices can often guide curriculum reform as they become a mediating process between teaching and learning. When they coalesce and complement each other, the three processes help to create a caring curriculum that is more relevant and meaningful to all learners. In this light, a valuable connection can be made to the productive pedagogy referred to as ‘recognition of difference’ (Education Queensland, 2002). In acknowledging the need for inclusive practices, our approach to assessment becomes a vital aspect to include in any professional conversations about pedagogy.

A framework for describing pedagogy as conversation

To illustrate my conceptualisation of pedagogy as conversation, I have revisited a framework that I developed after observing and talking to six classroom teachers as they taught mathematics to primary school children (K–6) over an eighteen-month period. The theoretical and methodological perspectives for that study are described in more detail in Smith (2000). In reconceptualising the pedagogical framework, the underlying theme of conversation was clearly evident, although I had not explicitly named it as conversation. One of the most salient aspects of the framework for this paper was the way in which opportunities for learning easily led to opportunities for assessing learning.

Placing assessment in the pedagogical picture

It is important to note that the teachers I worked with to develop this framework did not consider themselves to be exemplary or expert in their field. They saw themselves as ‘just trying to do their best’ and ‘trying to make a difference to the way kids see maths’. The focus for the previous study was initially to develop teaching strategies that would promote students’ thinking and understanding, in order to address the Working Mathematically outcomes that had been introduced into New South Wales schools (Board of Studies, 1997). What soon became obvious to the participating teachers was

that by changing their teaching strategies, they then had to change the way they assessed their students' progress.

The assessment strategies that the participants were using (mostly pen and paper tests) no longer provided enough evidence to determine the extent to which students were able to verbalise, clarify and record their thinking about mathematics topics. As work samples replaced worksheets, a new form of learner centred assessment became evident. Figure 1 represents the pedagogical framework that emerged from the data collected in the classrooms of the six participating teachers. The framework provides a visual representation of the relationship between the processes of teaching, learning and assessing in mathematics classrooms.

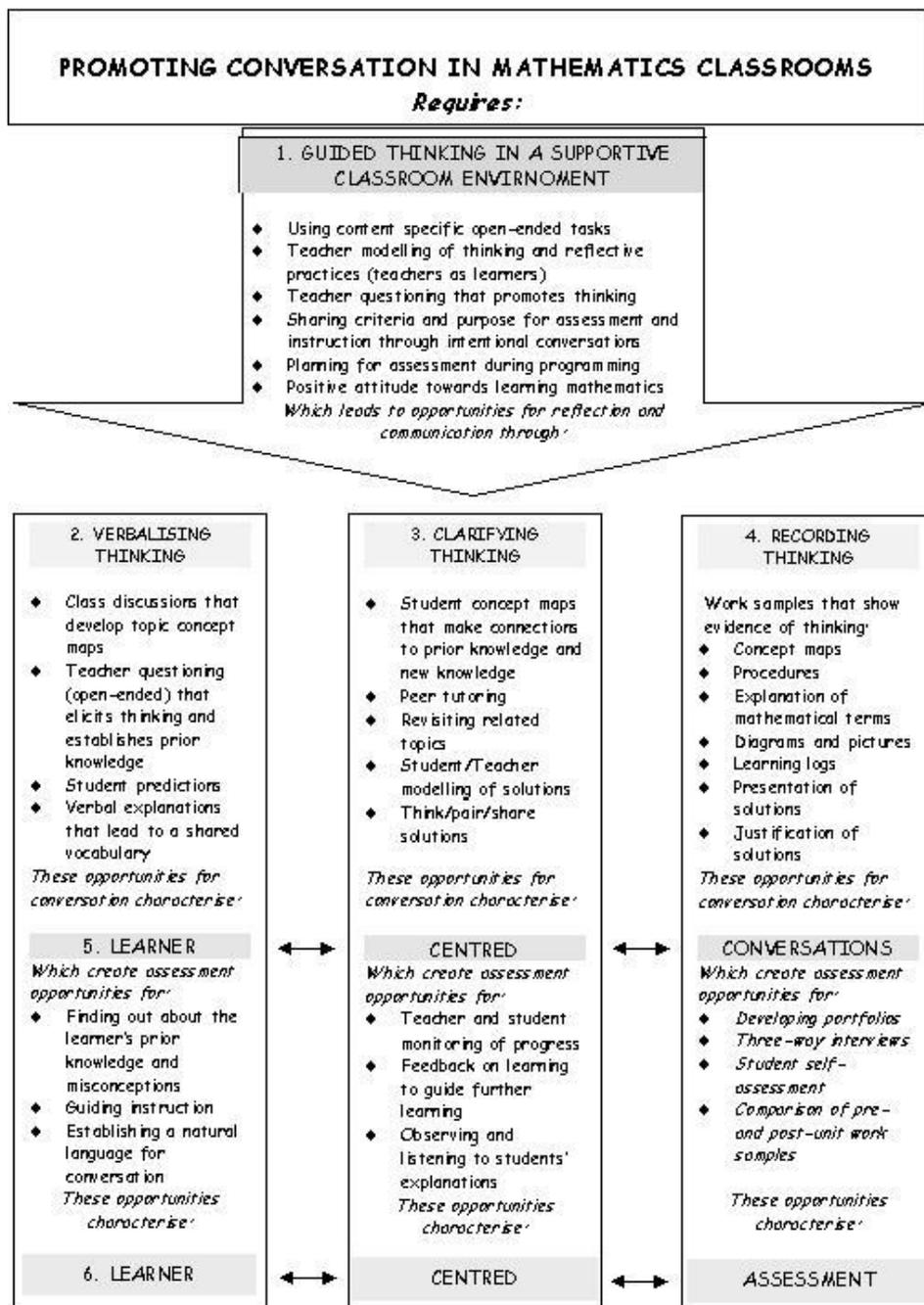


Figure 1. A pedagogical framework for promoting conversation in mathematics classrooms.

I now see that each of the pedagogical practices named in the framework promoted sense making through conversation. The use of the word conversation to conceptualise this approach is deliberate. Conversation suggests talk within a community of learners where there is a commitment to listening and learning from each other. Through conversation, understanding emerges as a shared construction. Such a two-way process provides pedagogical pathways that allow for *listening more* and *telling less*. The pedagogical pathways named in the framework explicate learner centred conversations that provide opportunities for learner centred assessment, which emerge from, and are embedded in conversations. The six pathways are:

1. Guided thinking in a supportive classroom environment;
2. Verbalising thinking;
3. Clarifying thinking;
4. Recording thinking;
5. Learner centred conversations; and
6. Learner centred assessment.

An important distinction needs to be made between aspects of conversation. As learners verbalise their thinking, they are participating in shared, *outer conversations* that have the potential to elicit and model a shared language, or metalanguage that can provide a tool for producing *written conversations* as students record their thinking. On the other hand, *inner conversations* take on a more metacognitive nature as learners reflect on and clarify their thinking so they can take personal ownership of their learning.

Guided thinking in a supportive classroom environment

The first pathway relates to the teacher's role as a facilitator of learning where the emphasis is on supporting learning through guided practice within a community of learners (Brown, Ellery & Campione, 1998). It views the teacher as a model and 'fellow player' in the learning process (Clarke, 1997) and provides a role description for teachers who are seeking to increase learner-centred conversations (both oral and written) that lead to learner centred assessment. Using content specific open-ended tasks (Sullivan, 1999) that create opportunities for classroom conversations requires students to think, reflect and communicate mathematically in a way that makes sense to them personally. Guided thinking in a supportive classroom environment acknowledges the affective domain of learning, and highlights the importance of positive attitudes, risk taking, sharing the purpose of learning experiences, motivation, and viewing mistakes as opportunities for learning. This aspect of the framework parallels the dimension of productive pedagogies referred to as 'supportive classroom environment', and the use of problem-based, open-ended tasks reflects the 'connectedness' dimension of productive pedagogies (Education Queensland, 2000).

Verbalising thinking

The second pathway incorporates all aspects of classroom practice that promote conversations in order to bring thinking out into the open. The emphasis is on

communicating meaning through *outer conversations*, or acts of reasoning that develop taken-as-shared understandings (Yackel & Cobb, 1996). It subsumes elements of oral conversation as a whole class using strategies such as concept maps to establish a shared vocabulary, teachers' use of open-ended questions to begin conversations and elicit prior knowledge, and other opportunities for active engagement in classroom conversations. This conversational process is crucial to the framework because it focusses on eliciting and modelling the natural language used by students to explain concepts. This promotes rehearsal and practice of language that in turn can be used to assist students to record their thinking using *written conversations*. The literature on productive pedagogies refer to this as developing a 'metalanguage' which is an important aspect related to the dimension of 'intellectual quality' within the productive pedagogies model. From a social perspective, it can contribute to allowing equal access to explicit mathematical goals and language.

Clarifying thinking

The third pathway focuses on more metacognitive processes such as students reflecting on, and monitoring their own progress. I have reconceptualised this metacognitive focus as opportunities for *inner conversations with the self*. These inner conversations can become part of outer conversations as students gain confidence to share their thinking in more collaborative and social conversations. Conversational strategies such as think/pair/share, student-created concept maps, as well as teacher and student modelling of think-aloud strategies and solutions were identified in the study as effective ways to clarify student thinking and reinforce conversation and reflection.

Importantly, teachers in the project also planned opportunities for students to revisit their work samples so they could clarify and reflect on their progress and make changes when they thought it was necessary. The practice of revisiting work contributed to dispelling the myth that mathematical processes and decision making must be quick or even instant, and cannot be wondered about in conversation. All of these strategies provided opportunities for self-assessment during the clarifying process. By articulating their thinking in collaborative conversations, students often corrected themselves as they explained and monitored their solutions. Acknowledging the importance of inner conversations places a strong emphasis on recognising individual differences in the interpretation of mathematical concepts and leads the way to recording understanding in more meaningful and contextual ways.

Recording thinking

The fourth pathway highlights the importance of students' written representations of thinking. This element refers to the need to collect evidence of students' thinking through authentic work samples that are naturally derived from learning experiences. The work samples were the result of students recording their solutions and responses to content specific open-ended tasks that further developed their metacognitive processes (inner conversations) of reflection and reasoning of solutions. I now see these written recordings as another form of conversation between learners in a classroom. As younger learners (students) record their thinking and their solutions to tasks, they provide an opportunity for more capable learners (often the teacher, but sometimes

their peers) to share their inner conversations. In turn, this can provide more focussed written and oral assessment and feedback, or outer conversations. In other words, written conversations (recording thinking) can create opportunities for meaningful learner-centred conversations that characterise the notion of shared and negotiated goals for learning. The three pathways of verbalising, clarifying and recording thinking can accommodate the ‘recognition of difference’ dimension in the productive pedagogies model as learners have opportunities to ‘author their own learning’ (Carpenter & Lehrer, 1999).

Learning to let go

Each of the pathways in the framework shares an underlying theme. The term ‘learner’ has been deliberately chosen to reinforce the idea that both students and teachers can walk these pathways as learners. Rather than place students and teachers in a dichotomous position of us and them, I see teachers and students working in synergy as co-learners. The conversational opportunities that this framework describes represent opportunities for co-learning in a supportive learning community. When *we listen more and tell less*, using a two-way conversational approach, we learn to let go of the perception that the teacher is the one who knows all. We learn to let go of preconceived notions that there is only one correct method in mathematics (usually the one the teacher models on the board), and that decisions in mathematics must be instant. The pathways present opportunities for recognising differences in the way we think about mathematics and how it may be relevant to our individual situations. It opens the pedagogical door wider to describe how assessment opportunities can naturally emerge from learning opportunities when we think of *pedagogy as conversation*.

Instead of being an afterthought, assessment needs to become a vital aspect of any professional conversations related to productive pedagogies. Placing assessment in the pedagogical picture has the potential to broaden the metalanguage of professional conversations so that teachers of mathematics can continue to imagine and describe how teaching could be ‘otherwise’. *Listening more and telling less* is all about learning to let go of the erroneous boundaries between teaching, learning and assessing.

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Finding the wave: The search for excellence*

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Mathematics educators continually seek for the best way of teaching mathematics – the big wave. They have discovered that, just as there is in surfing, there are many factors to be considered and have still not reached a definite or satisfactory conclusion. For every theory that is espoused there seems to be another that runs counter to it. This paper will explore some of the major trends in mathematics education in the past few decades, as well as the advantages and disadvantages associated with each one. An attempt will be made to examine the components that are required to prepare teachers for this puzzling task with appropriate analogies to the wave where possible. The results of research will be incorporated into the presentation.

Surfers travel the world looking for the ‘big wave’. Their world forms a sub-culture wherever they go. They confer, they hone their equipment, they practise their skills, they observe and make decisions. Finding the wave is just the beginning. They then need to work out the best strategy for riding that wave, to overcome it, to subdue it, to gain the prize, to feel the exhilaration of it all. The bigger the wave, the greater the thrill!

Mathematics educators travel the world of knowledge, ideas and research to find the ‘big theory’ – the one that will ensure excellence in teaching and learning mathematics. They too confer, they constantly seek to hone their equipment, they practise their skills, they observe and make decisions. How many, though, really think they have found the best strategy for maximum success in mathematics? Mathematics education researchers and educators seem to arrive at a theory, and observing that it works for their immediate environment, assume it will work for all environments. Just as surfers find differences in the surf in different parts of the world, so mathematics educators will find differences in students, even in very confined environments. The question arises, then: is there one best way of teaching mathematics? Or of learning mathematics? Is there one way for an individual or can different strategies work better at different times?

* This paper has been subject to peer review.

Waves

Waves are rather beautiful if you are standing well back on the beach or on some elevation. They look so magnificent, so strong, so rhythmic — and they are. From a small boat in the midst of the sea, though, the waves look altogether different. They look menacing, relentless, dangerous — and they are. So where we are determines, to a certain extent, how we feel about them. Wherever we are, however, we can feel in awe of them.

Mathematics is rather beautiful if we stand back and reflect on the magnificent achievements of great mathematicians or on the intricacies of mathematical thought. It is still beautiful if we have been able to achieve success in doing mathematics. If we have not, however, mathematics is like those waves when we are in a small boat in their midst: terrifying. Our search for excellence in teaching and learning, then, must take into account both those who can succeed and those who think they cannot succeed in mathematics.

There are various ways of referring to waves. We look for waves, we ride waves, we make waves. In mathematics education, we want to do all three. We want to find the right theory or theories, we want to use it to some purpose, and we want to succeed and make waves in mathematics. Finding the right theory is the initial step and, in many ways, the key to the process. The right theory is elusive and needs to be tested universally if it is to be applied universally. Is there such a thing? Is there a single theory that fits all cases? Dienes was one theorist who took an eclectic view of mathematics education and developed a theory of teaching mathematics with elements that he had gleaned from other theorists. Is this perhaps a better way to go? Is this the 'big wave' in mathematics education? Or is the 'big wave' in mathematics education just an ability to adapt to the situation whatever that situation might be?

Factors that count

For surfers, the critical factors are the surf and the waves, their equipment and their own skill and knowledge. Surfing is no fun if there is not a good surf or if it is so rough that it is unmanageable. The equipment surfers need are their boards, their wetsuits, their fitness, their understanding of the sea and themselves. For the mathematics educator the critical factors are their curriculum, their equipment which is both abstract and concrete, and their own skill and knowledge. The knowledge and understanding teachers need is their knowledge of mathematics and pedagogy in mathematics, their knowledge of their students and how they learn. How do we measure up? As we search for excellence, how well equipped are we? When reviewed, this statement seems to resonate with the AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* (2002) that categorises the standards in three domains: professional knowledge, professional attributes and professional practice. In this paper the elements of each of these might be organised differently but they are all there.

The curriculum

Mathematics curricula are basically determined by educational systems. We could call this the *intended* curriculum. Teachers have the responsibility of interpreting the syllabus in mathematics and do not necessarily interpret it the same way as the curriculum writers. Consequently they present what has become known as the *presented* curriculum. Students, being unique individuals, do not necessarily grasp what is presented in the way intended and so, they relate to the *received* curriculum. In one sense mathematics teachers have little control over the curriculum. They do have some responsibility for what is presented and it has been found that a collegial approach to the curriculum can help to align it more with the *intended* curriculum. Again, however, the teacher does not have complete responsibility for what students receive in the *received* curriculum. The students themselves should bear some responsibility. The teacher, however, needs to be alert to the different interpretations students might make and continually check on the students' understanding. They also have the responsibility to so prepare that what they present is less open to misinterpretation. Students' responsibility is first to be willing to listen and to contribute to the classroom experience, and to question anything not clearly expressed. It is a combined effort that counts, not just among teachers but also between students and teachers. Just as surfers are safer if they venture into the sea when others are around, so, too, teachers and students are safer if they work together.

Equipment

Teachers of mathematics, being already equipped with the appropriate syllabus, need to ensure their other gear is in good order and that they know how to use it. A novice surfer will find him or herself frequently 'dumped' by a wave if s/he has not learnt something about waves, currents, winds and sand. A surfer also needs some knowledge of angles, observational and problem solving strategies to make it to the beach successfully. Teachers of mathematics need to know the mathematics they are to teach, possess a repertoire of instructional strategies and materials — both concrete and print — and an understanding of the ways in which the students in the class learn mathematics best. In addition to all this, the teacher of mathematics needs to recognise that their equipment also includes themselves: their personality, knowledge, and their ability to enthuse their students and to develop appropriate relationships with them.

Knowledge of mathematics

It is unfortunate that the emphasis on process in mathematics syllabuses stressed in the past two decades has, in some cases, resulted in a decreased emphasis on the importance of teachers' knowledge of the discipline. It is heartening to see some trends that indicate that mathematics educators are moving to a more balanced viewpoint. The writer adapted a test of pre-service primary teachers she gave in 1964 to be appropriate for the current 1989 NSW Mathematics K–6 Syllabus and gave the revised test to a group of pre-service primary teachers in 1994 (Southwell, 1995). The results were quite alarming and seemed to indicate that teacher trainee knowledge of the mathematics they were to teach had deteriorated considerably. When one considers

other factors, however, the results are less alarming. For instance, the original sample was taken from the only three colleges on the NSW seaboard and in 1964 it was an elite group of students who went on to higher education. The 1994 sample was entirely different in that it was comprised mainly of first generation higher education students from what is usually regarded as a disadvantaged part of the state. Also, no matter how careful one is, it is impossible to revise a test adequately after thirty years: the world is a different place, mathematics education has moved on and emphasised different aspects of the discipline — that, of course, does not necessarily make it better.

Ma (1999) investigated teachers' understanding of mathematics in China and the United States of America. She concluded that, 'teachers who do not acquire mathematical competence during schooling are unlikely to have another opportunity to acquire it' (p. 145). It appears that most teacher education programs concentrate on how to teach mathematics rather than on the discipline itself. As students are individuals, and in any one class there are likely to be many levels of understanding and background, teacher educators have the challenge of handling both the mathematical deficiencies of each student and their level of skill in teaching.

Fennema and Franke (1992) affirm that, 'the belief in the importance of mathematical knowledge is shared by scholars in the field' (p. 148) and quote several mathematics educators including Ball (1988a, p. 12) and Post, Harel, Behr and Lesh (1988, pp. 210, 213). Ball's statement is particularly pertinent. She says, 'Knowledge of mathematics is obviously fundamental to being able to help someone else learn it.' Many of these very strong statements have been overlooked to the detriment of the state of mathematics teaching and learning.

There is some evidence, however, that mathematics knowledge is gaining greater emphasis than it has had for some years. The AAMT *Standards* is one of the most recent documents to say that 'excellent teachers of mathematics have a sound, coherent knowledge of the mathematics appropriate to the student level they teach'. The NCTM *Principles and Standards for School Mathematics* (2000) states that, 'Teachers need different kinds of mathematical knowledge — knowledge about the whole domain; deep, flexible knowledge about curriculum goals and about the important ideas that are central to their grade level' (p. 17).

Pedagogical strategies

Another standard provided by both AAMT and NCTM is that teachers need to know how to present mathematics to the students and have some understanding of the difficulties that students might encounter in doing mathematics. One is reminded again of Ma's (1999) assertion that unless students learn mathematics while they are at school, they may not have the opportunity to do so before they are required to teach it. In one sense this puts them in the uncomfortable position of not being sure if they can cope, and also in the enviable position of actually understanding what it feels like to lack knowledge and to struggle with it. This latter position is only a positive one if they have been guided into understanding themselves.

The *Cockcroft Report* (1982) advocated six types of instructional or pedagogical strategy for use in the presentation of each mathematics topic. These still have currency

and while individual educators might recommend different approaches, the basic truth is that no one method will suit everyone. The utilisation of the six Cockcroft types or styles will provide the best opportunity for students to receive the right treatment from at least one of them. Gardner's (1986) multiple intelligences is perhaps a more refined theory requiring the teacher's knowledge of the learning style of each student in the class. Using the Cockcroft approach, however, while it seems more 'hit and miss', on balance it has a greater chance of presenting material in a preferred way. Becker and Selter (1996) advocate an open-ended approach to teaching and learning mathematics (pp. 526–529). By this they mean the use of open-ended problems in an approach that gives students experience in finding new ideas. This is the result of combining knowledge and skills that the students themselves bring to the problem.

Instructional materials

The use of concrete materials in the teaching of mathematics has been fairly common for the past fifty years, though it was not always on an individual basis. It is somewhat disturbing to observe that once again, some teachers and students are thinking of concrete material as being only suitable for the younger children, say from 0–8 years of age.

This trend is to be deplored because some adults need concrete materials to understand some ideas. Certainly, there should not be any denigration of the use of concrete aids to assist students in their understanding of mathematical concepts.

The way in which the materials are used is important. Some of the most successful uses of concrete materials in the secondary school, for instance, is in collaborative activities that require some equipment; e.g. investigating the number of Year 8 students that can fit into a cubic metre.

Knowledge of student learning

As mentioned above, students are individuals and have their own unique ways of doing things. If teachers are to develop good professional relationships with their students, they will need to have some general knowledge about the learning of mathematical ideas. They will also need some specific knowledge about each of the students in their class.

Understanding of students' learning has changed over the past fifty years. The behaviourist approach of the first half of the twentieth century gave way to a more cognitive approach spurred on by the writings of Piaget based on his experiments with children. Piaget's stage development theory was accepted as a revolutionary idea and the notions of conservation, adaptation, assimilation and accommodation became well known among mathematics educators. Other theorists developed their ideas around the same time. Bruner suggested there are three levels of representing any particular concept, emphasised the structure of mathematics, discovery learning and introduced the spiral curriculum. Skemp and Gagne developed stage theories of another kind. Skemp said that one cannot acquire a concept unless s/he has acquired all the concepts on which it depends. Gagne wrote in term of prerequisite skills in a formally structured and validated hierarchy. Dienes used his eclectic approach and developed four rules for

mathematics education. At the same time but unbeknown to the western world, a young Russian by the name of Vygotsky developed his idea that learning is a social process in which the learning was most effectively achieved when interaction with others and particularly a more mature peer or adult took place. He applied the notion of scaffolding to this peer or adult support and introduced the 'Zone of Proximal Development' or ZPD. In this theory, there is a zone through which a learner can progress with the support of the more mature peer or adult to reach a higher level of learning.

A further major development took place when von Glasersfeld (1991) decided that constructivist theory was applicable to the learning of mathematics. The majority of teachers seem to be attempting to implement this style into their teaching. Science education also espoused constructivism though there are some notable people who now query its use.

Von Glasersfeld referred to Piaget, Skemp and Vygotsky as constructivists because they were able to change the emphasis from the teacher to the student and allow the students to construct their own knowledge of mathematics as a result of their experiences in the subject. He differentiated between radical and social constructivism and attributed radical constructivism to Piaget and social constructivism to Vygotsky because of the relative emphases on social interaction.

In the mind of the author, each new theory does not displace previous ones. It adds to the richness of the variety of approaches that are possible and, as is stated in the NCTM's *Principles and Standards*, 'there is no 'one' way' (p. 18).

The teacher

The AAMT *Standards* list three professional attributes for excellence in teaching mathematics and this is where it becomes very personal indeed. They fit nicely together and if the first one is evident, the others should follow. The three attributes suggested are personal attributes, personal professional development and community responsibilities. The statement about personal attributes cannot be stressed too highly. This is what it says:

The work of excellent teachers of mathematics reflects a range of personal attributes that assists them to engage students in their learning. Their enthusiasm for mathematics and its learning characterises their work. These teachers have a conviction that all students can learn mathematics. They are committed to maximising students' opportunities to learn mathematics and set high achievable standards for the learning of each student. They aim for students to become autonomous and self directed learners who enjoy mathematics. These teachers exhibit care and respect for their students. (AAMT, 2002)

The words that jump out of this statement are 'engage', 'enthusiasm', 'conviction', 'all students can learn', 'autonomous', 'self directed', 'enjoy', 'care' and 'respect'. Some causal relationships can be developed from these. Teachers exhibit care and respect for their students through believing they can do and enjoy mathematics, by presenting their subject enthusiastically to engage them and to lead to self directed and autonomous learning. Another way of putting this is that if teachers of mathematics

love and enjoy their discipline they will be keen to share their enjoyment with their students so will present appropriate mathematics to all students in a stimulating and engaging manner. These thoughts could be pursued further because the ideas behind them are so powerful.

Catching that wave

Your eye is on it, you anticipate as a result of your previous knowledge and experience the right time to move — you move and experience the fascination and thrill of the wave. If you lose your concentration, however, disaster strikes. So it is with teaching mathematics. If we know the subject, we can anticipate the strategies we will use and the students' responses. We will know the right time to move, and if our equipment is well cared for and our preparation thoroughly done, we can roll on with great enthusiasm. We become aware of the needs of each student and support them accordingly. We help them to develop the desire and the skill to learn.

Making waves

The theme of the conference is *Mathematics — making waves*. Mathematics can only make waves through the people who use mathematics. In the school context they are the students and the teachers. We need to ask ourselves: are we game enough to look for and explore the big waves and to be willing to cause at least a few ripples ourselves?

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Do you need to think to ride the wave?*

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When you catch that wave do you just let it take you where it will, or have you thought about what might happen and how you can guide your board to the beach? This is probably another way of saying, 'How much mathematical thinking is involving in riding waves?'. Is mathematical thinking important anyway? This paper will explore this issue and the development of thinking in mathematics. Strategies that are designed to not only develop mathematical ideas but also to use them will be considered. The results of research will be incorporated into the presentation as will practical suggestions and activities for the classroom.

Waves are fascinating phenomena. To the layman they are unpredictable, treacherous, majestic, relentless, overwhelming. They come in various sizes, from different directions and in varying strengths. They can take you to the heights in exhilaration or bring you to the deepest trough imaginable — a bit like life, really. A bit like mathematics, too, it seems.

How do these three things fit together: waves, life, mathematics? Do you need to think to catch that wave and ride it to the beach? Do you need to think mathematically to catch that wave and ride it to the beach? I have it from reliable sources that you certainly do need to think. My source says that you have to be thinking and thinking mathematically all the time. Besides the waves you can see and the currents you expect, there is always the chance of the rogue wave, the one that really tests your knowledge and skill. There is the dumper that throws you unaware into a new situation. Being able to analyse the situation, make decisions and apply appropriate action could save you from injury. So too with mathematics. The development of knowledge and the process of mathematical reasoning in all branches of mathematics could well contribute to the perfect job or to a sense of satisfaction or excitement.

Is life like this? Do we have to think all our waking hours? Some people even think during their non-waking hours, according to the Gestaltists. If we want to make good decisions, we do need to think at least most of the time. This thinking takes various forms. It might be recalling some incident or person. It might be explaining some idea,

* This paper has been subject to peer review.

justifying our actions or statements, making and testing conjectures, presenting sound arguments and/or proofs, or simply reflecting on the state of the world.

How do we learn these processes of thinking? This is where mathematics and other disciplines come in. These processes of thinking are the very processes that are facilitated through problem solving, and mathematical investigations in particular, and also through other aspects of mathematics, such as geometrical proofs. Thinking in itself is not sufficient. It is more useful if it can be applied in some situation. Surfers do this: they watch the wave, they are aware of themselves, they make decisions about when the wave is worth riding, when to bend and when to turn – to change direction.

In the processes of problem solving and mathematical investigations, it is well nigh impossible to separate concepts. Mathematical ideas are interconnected in functional contexts by their very nature. Reasoning is involved in developing strategies and processes that explain, check and support thinking. Questioning can often clarify situations and enable students to follow logical arguments or develop conjectures to be justified. Language and other forms of representations are needed to ask questions, explain and justify. Generalisations result from reflection.

In a statement setting out the goals for mathematical literacy, Romberg (1994) includes the areas:

1. *Learning to value mathematics:* Understanding its evolution and its role in society and the sciences.
2. *Becoming confident of one's own ability:* Coming to trust one's own mathematical thinking, and having the ability to make sense of situations and solve problems.
3. *Becoming a mathematical problem solver:* Essential to becoming a productive citizen, which requires experience in solving a variety of extended and non-routine problems.
4. *Learning to communicate mathematically:* Learning the signs, symbols and terms of mathematics.
5. *Learning to reason mathematically:* Making conjectures, gathering evidence, and building mathematical arguments. (p. 288)

Each area contains some degree of mathematical thinking or reasoning. The process of reasoning in its broadest sense is what needs to be explored. To do so, some of the processes involved in reasoning are explaining, justifying, generalising, argumentation and proof although there are others that are also relevant such as analysing, representing, visualisation.

Explaining and justifying

A Year 2 class was given the following exercise, and from this Whitenack and Yackel (2002) were able to explain the distinction between explaining and justifying. They also explained the importance of these processes in reasoning and hence in developing mathematical arguments. The children were asked to solve this problem:

Aunt Mary has 31 pieces of candy on the counter and Uncle Johnny eats 15 pieces of candy. Show how much candy Aunt Mary has on the counter now.

One contribution and was written on the board thus: $30 - 15 = 15$, $15 + 1 = 16$ and then explained:

Um, I took that 1 away from the 31. And 30 minus 15 equals 15. And plus 1 equals 16. If you take that 1 and add it onto the 30 to make 31, it's just when you minus you just have 1 higher number, and if you take the 1 off the 30, add onto the 15, you get 16. And that's where I get 16.

This was not easy to follow for some students so the teacher asked why the first student solved the problem that way. The student, by way of justification, pointed out that he had used the doubles fact that $15 + 15 = 30$ and that meant he was able to arrive at a result more easily. The difference between explaining and justifying could be described by noting that explaining tells *how* a process is completed while justifying indicates *why* it will work.

Argumentation

The process of argumentation links these two processes of explaining and justifying. The *Principles and Standards for School Mathematics* (NCTM, 2000, p. 56) says that 'instructional programs should enable students to develop and evaluate mathematical arguments and proofs'.

Arguments can be considered as other than fairly negative experiences but as Whitenack and Yackel point out, they can be beneficial as far as mathematics is concerned. They advocate argumentation for several reasons, namely:

- To explain or justify their results, students need to think over their solutions and evaluate them. An even stronger mathematical argument or even a better way of reaching the solution could result.
- Students have the opportunity and are challenged to think about new mathematical ideas.
- When used effectively, mathematical argumentation has benefits for both the individual students and for the whole class. (p. 525)

Asking questions is fundamental to good argumentation and it is a skill that needs to be developed as students seem to lose their ability to ask good analytical questions once they leave kindergarten. An example is useful at this point:

Using the school playground or a stimulus picture of the school playground, what are some of the mathematical questions we could ask to help us use it better?

Having students pose questions about mathematical situations is one way of encouraging them to investigate mathematical situations. Manouchehri (2001) uses problem solving as the first phase of a four-point instructional model. In a sixth grade class she makes a comment of the day, e.g. even numbers, and asks the class for good questions about them. The students are given ten minutes to write down as many questions as they can, and are warned they will need to share them with their peers. A list of good questions is compiled on the chalkboard from which students are asked to

select the one they want to work on with others. The final groups may be of different sizes. Small group problem solving, then large-group discussion follow with a final sharing in Phase Three. Further problems and projects are considered in Phase four. This provides a logical sequence of action and provides the opportunity for students to not only pose problems but to share their explanations and justifications.

Mathematical argumentation often results from conjectures and testing conjectures in problem posing. Some educators claim that such thinking is not possible for primary aged students. Reid (2002) tested this with fifth grade students. He used a pattern of reasoning, involving conjecture or rule, test that rule, then either use it for further exploration, reject it or modify it. He cautioned that first 'we must be clear as to what we mean by "mathematical reasoning"'. Secondly, we need to consider that different forms of reasoning are used by students and thirdly, we 'must consider not only these elements of reasoning, but also the overall pattern of reasoning' (p. 27). Sternberg (1999) claims that mathematical reasoning requires analytical thinking and creative and practical thinking as well (p. 43). This suggests that learners may have certain skills such as analytical thinking but unless they can apply their thinking, they are not really reasoning mathematically.

How do we develop arguments in mathematics? Whitenack and Yackel (2002) claim the critical element is the environment in which the students are operating, so the question becomes: How do we develop environments that foster mathematical arguments? Knowledge of the students' strengths and weaknesses and understanding of the roles of the teacher and the students in the classroom will help in this regard. Constant encouragement for students to present their questions and ideas is needed and students need to understand they have a responsibility in discussions, such as learning to listen to others, being willing to have their own ideas scrutinised by others, developing communication skills and respect for one another.

Stein (2001) emphasises the necessity of having a good task for the students to work on and the use of a variety of prompts to stimulate students' thinking. She recommends not challenging students with 'Can you explain that?' each time but to find other ways of checking meaning and understanding. The classroom atmosphere is another of Stein's emphases. Her approach, however, differs from some others in that the students are first given time to find their individual solutions, then are asked to share their answers and reasons for their answers with a small group. Certain students will be asked to present their answers and strategies to the class.

Time is needed to develop a helpful environment. Students' ability to engage in mathematical arguments takes time also and needs to be constructed gradually. As students solve problems, they need to ask themselves questions such as, 'Why is this true?' or 'How else could I have solved this?'. When they are ready, they will ask others the same questions and even start encouraging others to do likewise.

Generalisation

In a NSW DET *Curriculum Support* (1999), an explanation of generalisation is given and a link made with concept development. Sometimes teachers assume students are constructing concepts when they have no real understanding as to how they attain

concepts in the first place. This can lead to considerable confusion and frustration on the part of the teacher as well as the student.

A way of looking at concept formation is that it is likely to occur when we use our reasoning powers to abstract the common element from a series of experiences. If the experience is repeated often enough, the concept can be validated and consolidated.

It is not sufficient to perform a series of moves, as it were, to get an answer to a problem. There needs to be an attempt to interpret that answer and the process whereby it was reached in order to develop a method or process that could be used in solving related problems, i.e. problems that are isomorphic. The introduction of algebra can be based on the generalised ideas students gain through their investigation of patterns (Martinez, 2002, p. 329).

Russell (1999) suggests that even quite young students are able to generalise. She stresses four points about mathematical reasoning. They are first that:

mathematical reasoning is essentially about the development, justification and use of mathematical generalisations... Second, mathematical reasoning leads to an interconnected web of mathematical knowledge within a mathematical domain. Third, the development of such a web of mathematical understandings is the foundation of what I call 'mathematical memory', what we often call number sense which provides the basis of insight into mathematical problems. Fourth an emphasis on mathematical reasoning in the classroom, as in the discipline of mathematics, necessarily incorporates the study of flawed or incorrect reasoning as an avenue toward deeper development of mathematical knowledge (p. 1).

This provides a formidable consideration of the role of generalisation in mathematical reasoning. Russell does describe the links between the use of concrete materials and generalisation and in doing so, challenges some of the traditional thinking concerning development stages of children's learning.

Proof

Argumentation leads to proof. Waring (2000) specifies four levels of proof. These are subtitled *convince a friend*, *convince a penfriend*, *towards formal proof*, and *proof for all*. The development of proof may be a long process and there may be some students who never need to reach the final stage. The key idea seems to be that it is considered proof if the students are convinced, on challenge, that they are correct and can justify their results to their own satisfaction. Some of Waring's levels of proof can be likened to the processes of explaining and justifying. In one sense, explaining is the first level of proof for some children, while others will be able to justify their results, a second level of proof. An example is the angle sum of a triangle. It can be demonstrated and therefore explained using the strategy of tearing or cutting the three corners off the triangle and placing them along a line to form a straight angle. A more formal proof at the second level takes into consideration parallel lines and alternate angles. This can then be extended to quadrilaterals and other polygons.

Porteous (1994) defines proof in this way:

A proof of a statement is any adequate expression of the necessity of its truth (p. 5).

He assumes that as students mature and develop greater capabilities in mathematics, they will be looking for more formal proofs. This may be so but they probably do not often look for the most rigorous of proofs.

Flores (2002) inquired into the schemes that students in primary school used to justify their results. Some were purely externally-based schemes for justification such as 'Because Mom told me'. Others used tables and relationships between numbers. Some used their perception of the physical situation, i.e. the look of it, and others made models with concrete materials. Any analytical methods used were mainly based on counting.

Herbst (2002) introduces two aspects of proof: didactical contract and double bind. She sees that it is important not to treat proof only in the formal sense. The teacher's task is to devise tasks for the students that will encourage students to develop proofs. The idea of worthwhile mathematical tasks is the key. Wittmann (2001) also emphasises the importance of worthwhile mathematical tasks which he refers to as 'substantial learning environments' (SLEs). The criteria he uses to establish the authenticity of mathematical tasks are given as:

1. It represents central objectives, contents and principles of teaching mathematics at a certain level.
 2. It is related to significant mathematical contents, processes and procedures beyond this level, and is a rich source of mathematical activities.
 3. It is flexible and can be adapted to the special conditions of a classroom.
 4. It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for educational research.
- (p. 2)

Even if one maintains that proof is rigorous, it has been shown that small children can begin to develop the processes that will eventually become formal proof. Too often teachers are themselves uncertain of the processes and not sure of the best way to approach them. Quite small children learn to ask questions and they can be assisted to explain and even justify the things they do, the mathematical processes they carry out. For their level, they are reasoning in whatever mathematics they are doing.

Helping students to develop mathematical reasoning

Besides the activities mentioned above, there are some general strategies that can be helpful for some students. Bright (1999) suggests three basic principles under which several others can lie. The three principles are:

1. Learn mathematics differently. This could be the introduction of concrete materials or the use of the environment to represent mathematical concepts.
2. Study the curriculum materials.
3. Interact with students.

Co-operative learning, technology, and different forms of assessment are all strategies that would help students to reason mathematically. Ittigson (2002) advocates several

of these strategies, together with oral and written communication, as ways of helping students to become mathematically powerful.

Conclusion

Mathematical reasoning involves a number of mathematical processes which are interconnected and in some cases overlapping. Reasoning takes various forms and includes aspects of questioning, applying strategies, communicating and reflecting. The processes that contribute to these are explaining, justifying, conjecturing and testing conjectures, generalising and proof. Activities, discussion and investigations in the mathematics classrooms need to be selected or devised to develop these processes for students to engage successfully in working mathematically.

So, do not get dumped. Even better, do not be a dumper. Even rogue waves can be managed if we are prepared in every way.

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Reforming arithmetic in the primary school years: The importance of quasi-variable expressions in arithmetic relations*

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A mathematics curriculum for the primary school of the 21st century must provide a stronger bridge to algebra in the later years of school, and must also strengthen children's understanding of basic arithmetic. This presentation will look at some approaches to early algebraic thinking which can begin in the primary school. It will examine the potentially algebraic nature of arithmetic rather than moving students from arithmetic to algebra. It will report on some recent research, and discuss current attempts to rethink the study of arithmetic away from an almost exclusive focus on computation.

Why algebra and arithmetic need to come closer together

Why should a re-examination of the relationship between arithmetic and algebra be important for the reform of the school mathematics curriculum at this present time? There are several reasons why it is time to question the traditional divide between these two areas of mathematics. Arithmetic and algebra have different histories and have come into the current curriculum with quite different communicative structures. To put it at its simplest, arithmetic is characterised for the most part by procedural thinking; that is, its focus is on getting an answer, and in setting out the methods or procedures that allow one to reach a correct answer or to check an answer. On the other hand, algebraic thinking is often described as relational or structural (Kieran, 1992). Its purpose is to detect, identify and communicate generalisation and structure.

Secondly, arithmetic and algebra have played quite different roles in school education. Arithmetic was seen as an essential part of a compulsory primary education, and

* This paper has been subject to peer review.

The early section of this paper has been adapted from a plenary address by Max Stephens at the ICM 2002 Satellite Conference on the Reform of the Mathematics Curriculum and Education in the 21st Century at South West China Normal University, Chongqing, China, 17–20 August 2002.

algebra as appropriate to the high school, and even then only for those with the ability for abstract symbolic thinking. Both these assumptions are being questioned today.

Further arguments allow us to question the traditional divide between algebra and arithmetic. Various forms of numerical algebra are now emerging which challenge the place of traditional school algebra. On the other hand, arithmetic tasks and expressions have always retained a capacity to be considered as algebraic objects, especially when left in unexecuted form. In short, the traditional divide between arithmetic and algebra is no longer defensible in a school mathematics curriculum of the 21st century.

Impediments to early algebraic thinking

Carpenter and Franke (2001) point to a well established belief among researchers, ‘that children in the elementary grades generally consider that the equal sign means to carry out the operation that precedes it, and that this is one of the major stumbling blocks when moving from arithmetic to algebra’ (Kieran, 1981; Matz, 1982) (p. 156). Viewing arithmetic expressions as algebraic objects, especially when these expressions are left in *unexecuted* form requires what Collis (1975) referred to as a *lack of closure*. He identified ‘acceptance of lack of closure (ALC)’ as a key to thinking algebraically. Incidentally, he also viewed ALC also as an indicator of the Piagetian late concrete-operations stage which he believed was required for algebra. This argument has been used wrongly to justify the exclusion of algebra in any form from the primary school curriculum.

In the 1970s, tests prepared by the Australian Council for Educational Research (ACER) had items such as:

$$7 - 5 + \square = 7$$

$$746 - 263 + \circlearrowright = 746.$$

The first item can be done using guesswork or calculation to decide that 5 is missing. However, Collis reasoned that the second item becomes very difficult if done as a calculation because it needed a student to resist the urge to ‘close’ the $746 - 263$ to 483.

Carpenter and Levi (1999), for example, introduced first and second-grade students to the concept of true and false number sentences. One of the number sentences that they used was $78 - 49 + 49 = 78$. When asked whether they thought this was a true sentence, all but one child answered that it was. One child said, ‘I do because you took away the 49 and it’s just like getting it back’.

This example shows, in the first place, that young children, when they are provided with rich material, can engage in quite insightful algebraic thinking. Second, its focus is away from computation. Its goal is to focus children’s attention on the underlying mathematical structure exemplified by the sentence $78 - 49 + 49 = 78$. Carpenter and Levi were not asking children to perform a calculation of the numbers on the left side of the equals sign. Nor is their example tied to the particular numbers 78 and 49. We can refer to *this* use of numbers as quasi-variables, because it is one of a ‘family’ of number expressions exemplifying the same underlying mathematical relationship which remains true whatever numbers are used.

Numerical expressions and quasi-variables

Some other examples of quasi-variable expressions are:

$$312 - 123 = 313 - 124$$

$$546 + 234 = 545 + 235$$

An important feature of these expressions is that the equals sign stands for a relationship between the numbers on either side. It is not a command to carry out a calculation. Children have to accept the lack of closure in these arithmetical expressions. By being left in unexecuted form, the truth of the expression can be justified by examination of the relationship between the numbers. There is a good reason for applying the term ‘quasi-variable’ in cases such as these where a structural rule or relationship between the numbers can be used to generate other expressions whose truth can be justified by appealing to a generating rule for the expressions rather than by appealing to calculation. In the above two cases, the expressions are also independent of the particular number-base 10. Other examples may not be so obvious at first, such as the expressions shown below on the left-hand side:

$$312 - 123 = 423 - 234$$

$$312 - 123 = (312 + 111) - (123 + 111)$$

$$546 + 234 = 445 + 345$$

$$546 + 234 = (546 - 101) + (234 + 101)$$

The truth of these expressions can be checked using subtraction or addition of numbers in base 10. However, as the expressions on the right show, they are variations of: $a - b = (a + c) - (b + c)$ and $a + b = (a - c) + (b + c)$. However, knowing the rule and using it to generate new expressions are not dependent on knowing its formal expression.

Going beyond missing-number sentences

Working with quasi-variable expressions provides a counterbalance to the use of ‘missing-number’ sentences in upper primary and junior high school where finding the value of an unknown often dominates students’ and teachers’ thinking. As Radford (1996) points out, ‘While the unknown is a number which does not vary, the variable designates a quantity whose value can change’ (p. 47).

Through the use of missing-number sentences such as $\square + 8 = 23$, and $63 - \bigcirc = 49$, teachers think that they are introducing children to algebra. Later, these sentences will be expressed using literal symbols in forms such as $x + 8 = 23$, and $63 - y = 49$; but that is to limit algebra to dealing with unknown numbers. The use of numerical sentences representing quasi-variable expressions can provide a useful and accessible gateway in the early years of school to the concept of a variable.

Meeting and discussing quasi-variable expressions acquaints children with what may be called ‘algebra underground’. There is no obvious use of literal expressions. Quasi-variable expressions do not look like the algebra of high school; but the forms of mathematical discourse which these expressions illustrate and the kind of thinking needed to probe their truth are both truly algebraic.

Implications for the reform of primary school mathematics

This is not an easy task when teachers' vision has for so long been restricted to thinking arithmetically. In the primary school, this means attending to the symbolic nature of arithmetic operations. Research suggests that many of today's students fail to abstract from their primary school experiences the mathematical structures that are necessary for them to make a later successful transition to algebra. As Carpenter and Franke (2001) point out: 'one of the hallmarks of this transition from arithmetic to algebraic thinking is a shift from a procedural view to a relational view of equality, and developing a relational understanding of the meaning of the equal sign underlies the ability to mark and represent generalisations' (p. 156). Here are four suggestions:

- describing and making use of general processes and structural properties of arithmetic, generally — of quasi-variable expressions in particular;
- focussing on the notion of equality, and devising problems using the equality sign that help students to accept and to work with a lack of closure in arithmetical statements;
- generalising solutions to arithmetic problems that assist students to develop the concept of a variable in an informal sense;
- providing opportunities for students to discuss their solution strategies to these problems in order to highlight fundamental mathematical processes and ideas.

Some other examples

The hundreds chart

Number charts and grids provide a range of progressively more sophisticated explorations of number patterns leading to quasi-variables. The task shown below is to investigate the sum of the numbers in a three-by-three grid.

Some prior work on the sum of three consecutive numbers is assumed. In looking at the sum of three consecutive numbers, the aim is to have students see that the sum is always three times the 'middle number'. It will be important to have them try many examples; to record the sum using a number sentence involving 'is equal to'; to see if the pattern works in every case; to predict the sum without having to add all the numbers; to work backwards by deducing what the three numbers are if they know their sum; and to explain why they think this works. The goal is to have children justify their thinking, for example, by saying, 'I can always take one off the largest number and give it to the smallest number, and so the three numbers all become the same'.

In Figure 1, there are many different approaches that can be adopted in exploring the sum of the numbers 'inside the box'. One way of proceeding is to consider the three rows of numbers. The sum of the first row is 3×26 , that of the middle row is 3×36 , and that of the bottom row is 3×46 . Adding the three products leads to a 'grand total' of 324 which is 9×36 . This approach could be refined further by showing that the respective totals increase by 30 each time, so that, by 'taking away' 30 from 3×46 and 'giving it' to 3×26 , there would be three groups of 3×36 , that is 9×36 . So, the total of

the numbers in the three-by-three grid is equal to nine times the ‘middle number’. Students, should have opportunities to explore other three-by-three arrays; and even asked to create a three-by-three array when only the total of the nine numbers is known.

At this stage, there is no strong case for saying that quasi-variables are in use. But the introduction of quasi-variables is much more plausible in Figure 2 where students are simply given the middle number of a three-by-three grid (in a hundreds chart) and asked to deduce all the other numbers and the total of the nine numbers.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 1

				45					

Figure 2

Working with different middle numbers provides a clear instance of working with quasi-variables, since students have to attend to the structural rules for generating the full set of nine numbers. In Figure 3, there is a general number n to work from. Once again it is assumed that the three-by-three grid is part of an incomplete hundreds

chart. This task is suitable for students in the middle and upper primary years or the early years of secondary school. Students can also investigate what happens if the three-by-three grid is not part of a hundreds chart. They could, for example, investigate the sum of numbers in a three-by-three grid which was part of a month calendar of dates (Milton, 2002). Initially, the task could show all the numbers and ask students to find the 'grand total' of the nine numbers. Later, students could be given one number only in the grid. And finally, students could be given an unknown middle number n and be asked to generate the remaining numbers and the total.

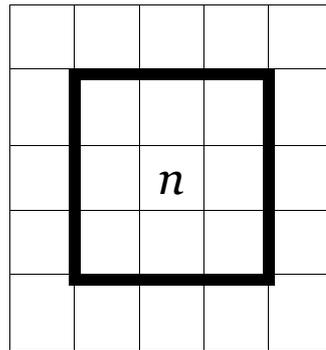


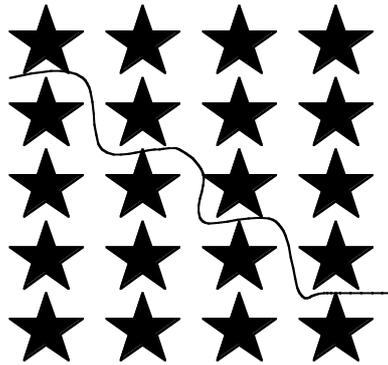
Figure 3

A geometrical illustration

Arithmetical entities and geometry can combine to give powerful illustrations of quasi-variables. The triangular numbers below have been 'doubled' to form a rectangle. A quasi-variable relationship involving the base and height of the rectangle generate the corresponding triangular number. This rule can also be expressed algebraically.

1st triangular number	★	1
2nd triangular number		$3 + 3 = 6$ $2 \times 3 = 6$ $3 = (2 \times 3) \div 2$
3rd triangular number		$6 + 6 = 12$ $3 \times 4 = 12$ $6 = (3 \times 4) \div 2$

4th triangular number



$$10 + 10 = 20$$

$$4 \times 5 = 20$$

$$10 = (4 \times 5) \div 2$$

From the above, the 5th triangular number 15 can be written as $(5 \times 6) \div 2$. Extending the same reasoning, the n th triangular number can be expressed as $n \times (n + 1) \div 2$.

Conclusion

Blanton and Kaput (2001) encourage teachers, especially in the primary school, to grow ‘algebra eyes and ears’ (p. 91) in order to see and use these opportunities. This is not an easy task when teachers’ vision has for so long been restricted to thinking arithmetically. In a mathematics curriculum for the primary school of the 21st century, teachers and students need to explore the potentially algebraic nature of arithmetic. This can provide a stronger bridge to algebra in the later years of school, and can also strengthen children’s understanding of basic arithmetic. Any reform of the arithmetic curriculum in the primary school must address these two objectives.

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Students' models, methods, limitations and assumptions for an applications task in a CAS calculator environment*

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Student responses to an applications task modelling trends in number of deaths due to HIV/AIDS infection in USA and Australia are examined. This task can be done completely using a graphing calculator or graphing software but the students for whom the task was designed also had access to CAS calculators which could have been used for support in algebraic manipulation. The students' models, their consideration of underlying assumptions and limitations in these, and their methods of producing their models, and their investigations of the properties of specific functions used by researchers to model AIDS data are presented.

Introduction

Mathematical modelling, applications, and the use of electronic technologies such as graphing calculators and computers have received increasing attention in curriculum documents in several Australian states in recent years at the upper secondary level (e.g. Board of Studies, 1999; Queensland Board of Senior Secondary School Studies, 2001). Computer algebra systems (CAS) have been available also for several years on computers but have had little impact on the secondary curriculum. The arrival of hand-held technologies incorporating CAS make their use in secondary schools feasible even at lower levels of schooling (Zehavi & Mann, 1999). The CAS-CAT research project in Victoria (<http://www.edfac.unimelb.edu/DSME/CAS-CAT/>) is investigating the effects of different CAS calculator environments on the nature, teaching, and assessment of a senior secondary mathematics course. At the three project schools, Year 12 students are enrolled in Mathematical Methods (CAS) for which they sit an external examination as well as school-assessed coursework. The focus of this paper will be an applications task (available from the CAS-CAT website) dealing with the mathematical modelling of the spread of HIV/AIDS in the USA and Australia. This task formed part of the students' school-assessed coursework.

* This paper has been subject to peer review.

Gellert, Jablonka and Keitel (2001) point out that the model in mathematical modelling ‘is by no means fully determined by the constraints of the situation’, rather the ‘mathematical concepts and methods at hand, the technical means (... calculators, computers), including rules for their use’ (p. 61), among other things, play crucial roles in which particular models are produced. Furthermore, they introduce the notion of ‘reflectiveness as a necessary competence for mathematical literacy’. As part of this reflectiveness they see students as needing to examine the models they construct for ‘appropriateness and reliability in a specific context’ by reflecting on how each model ‘explicitly accommodates the non-mathematical constituents’ (pp. 70–71) of the situation being modelled. Pollak (1997) also points out that, ‘the problem is not solved if the mathematics is perfect but too clumsy to be used in the real world, or if it results in unreasonable explanations and predictions’ (p. 93).

The study

In this paper the models, and assumptions and methods senior secondary students used to produce these, for an applications task in a CAS calculator environment will be investigated. Also, the limitations the students identified in using the models for various purposes such as prediction will be examined.

A total of 78 students sat for the applications task. At school A there were two classes ($n_1 = 12$, $n_2 = 18$) with the same teacher. These students had access to Casio ALGEBRA FX 2.0 calculators. At school B there were also two classes ($n_3 = 17$, $n_4 = 18$) but with two different teachers. These students had access to TI-89 calculators. At school C there was only one class of students ($n_5 = 13$) and these students had access to an HP-40G calculator. The conditions for completing the task were determined by the teachers. At school B, for example, the task was released in three parts, the first two being the focus of this paper. For the first lesson students were permitted to discuss the first part of the task with the person sitting beside them. Subsequently, only the teacher could be consulted for clarification. The students were allowed seven lessons on the task over one teaching week. Work was done in a log book which was collected at the end of each daily session. At the researcher’s request, the students were asked to indicate in their log books when they used their CAS calculators. The log books which included both exploratory and final working were subjected to intensive response analysis.

Results and discussion

Modelling US HIV/AIDS death rate data

Students were given a verbal description of the data, with some key numerical points, and asked to sketch a graph showing all information given about the number of deaths from HIV infection in the USA from 1990 until 1998. They then had to find an equation for a function that fitted the data for the number of deaths from 1990 to 1994, before the introduction of anti-retroviral therapy, and use it to predict the number of deaths in 1998 if the trends had continued unchanged. As this involved a linear function, it was

no surprise that all students produced a suitable model with ease and were able to use it for prediction.

As the effects of new treatments began to be reflected in the number of deaths during 1995, it was suggested as appropriate to model the trends separately before and after 1995. The students were therefore asked to find functions fitting the data from 1995 to 1998, firstly using polynomials and then any function predicting a limiting value of 3 deaths per year per 100 000 people in the long term. The data and information given were:

There was a slowing of the rate of increase due to the effects of anti-retroviral treatments beginning to take effect during the year 1994–1995. In 1994 and 1995, there were 16.16 and 16.35 deaths, respectively. From the end of 1995 to 1998, the number of deaths decreased, dropping quickly at first, then the rate of decrease slowed. In 1998 there were 4.90 deaths.

The limitations of using a polynomial function were to be discussed as well as underlying assumptions that needed to be made to allow its use for prediction.

Variety of models

A wide variety of quadratic models and two cubic models were produced for the polynomial models (see Figure 1). One student produced an inappropriate linear model while another suggested an hyperbola in addition to a cubic and a quadratic model. For the limiting value case students produced mainly hyperbola models but 8 students chose truncus models while 3 others chose exponential models (see Figure 2). In addition, one student used a trigonometric model. These models were relatively free of mechanical errors related to algebraic manipulation.

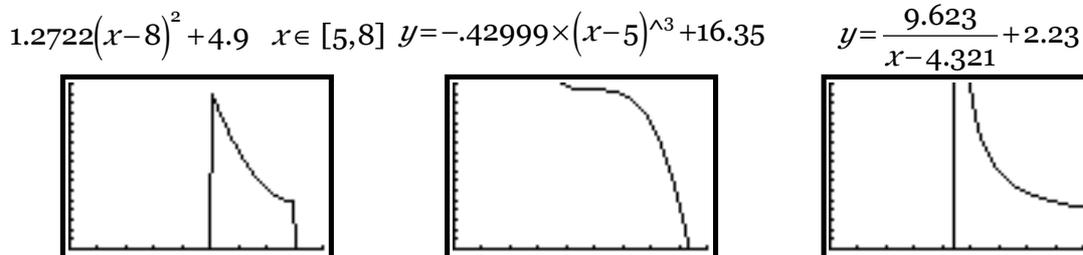


Figure 1. Models of the USA data 1995–1998.

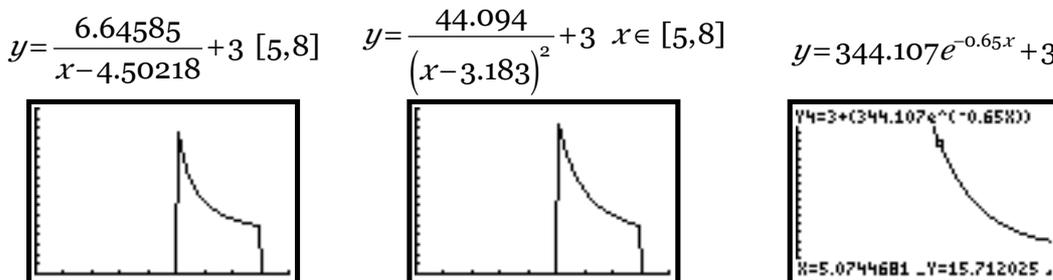


Figure 2. Limiting value models of the USA data from 1995.

Variety of methods

Quite a variety of methods were employed in generating the models as is obvious when the methods for producing quadratic models are examined. Twelve students used quadratic regression on their CAS calculators using 2, 3, or 4 points. Even though quadratic regression is also available on the TI-89, no students from school B used this calculator facility.

Various algebraic approaches were used to produce other quadratic models. The turning point form of the quadratic function was used by 35 students who assumed one of the given data points was the turning point, substituted for the other datum point and solved the resulting equation using hand calculation or define and solve on the TI-89 or possibly equivalent facilities on the Casio.

The solution of simultaneous equations was used almost exclusively by students from school A although other students from this school used the turning point method just described. Six students used simultaneous equations involving three points substituted into the general quadratic equation. The three points were two given points and a third that was ‘made up’ or read from the graph the student had sketched using the data and information given. Thirteen students used two points and the gradient at a point that was ‘made up’ or read or calculated from their graphs. These ‘made up’ values were reasonable estimates informed by students’ sketches of the given information and data.

Ab8: There is an assumption that there will be a turning point at an x value, and in order to use $\frac{dy}{dx}$ to find the function we assumed the gradient to be -0.1 at $x = 8$.

A further three students used one point and the gradient at two points. In all these cases students used their CAS calculator to solve the simultaneous equations. In addition, one student from school B used two points substituted into the general equation of a quadratic function and the fact that the value of the gradient function at the turning point is zero to generate three simultaneous equations that were then solved by hand.

One further quadratic model was merely stated by the student with no clues as to how it was generated.

Uncritical acceptance of models

At times students appeared to have been seduced by the power of the technological tool at their disposal into being uncritical of the models it produced. They appeared to be unaware of the need to bring declarative knowledge about curve fitting to their use of the calculator. For example, nine students chose the quadratic regression feature on the calculator to produce the model $y = -0.83x^2 + 6.69x + 2.28$ (Figure 3), but used only two points to fit the model, ignoring the fact that a multitude of quadratics will pass through two points. Perhaps, the fact that regression is seen as producing the ‘best fit’, enticed them into not considering other aspects of the given information that would have called their models into question. However, when asked to critique the model they were able to do so for the general quadratic form.

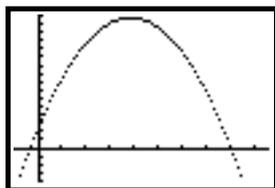


Figure 3. Quadratic regression model using only two data points.

- C5: For this equation to realistically represent the given situation the domain would have to be restricted, no negative x values as we are only interested in the years since 1990. The range would also have to be restricted such that the value $0 +$ negative values are not included – you cannot have negative deaths and there is very little chance of there ever being 0 deaths. With this particular model it must be assumed that the death rate continues to decrease.

Model validation

The majority of students ignored the need for validating their various models by testing how well the model holds with a set of data not used in model development.

- Ba14: My result is an exact result, when the points are checked using the value option - both points (0, 16.35) and (3, 4.9) are exact.

The points being referred to are the same points used to develop the model and thus necessarily lie on the curve. However, a few students realised the need for such validation choosing their data for model development very carefully to allow them to still have real data with which to validate their model.

Using the models

Students were asked to use their models for the HIV/AIDS data from 1995 onwards to determine two estimates of how much the new therapy reduced the likely total number of deaths from 1995 to 1998. Again, they had to identify assumptions made to arrive at a result, explaining whether this was an under or over estimation or exact. They were also asked to explain what this reduction was (e.g. if it was it lives saved), quite a subtle interpretative task. Students at school A did not do this part of the task so the following analysis is based on the responses of the remaining 48 students.

An appropriate conceptualisation of how to measure the effect of the new therapy proved to be quite challenging for the majority of students as it relied on a full understanding within the non-mathematical task context of the three mathematical models (e.g. those in Figure 4) they had produced earlier. An acceptable method of solution involved extrapolating from the linear model to find the death rates per 100 000 for the years 1995–1998 and adding these to give the total death rate in this period if the therapy had not been introduced and the previous trend continued. Similarly, the total death rates for the same period can be calculated using the polynomial model and then the limiting value model. Finally, the reductions for the two

models of the trend once the therapy was introduced are calculated by subtracting the total death rates for these models from that for the linear model.

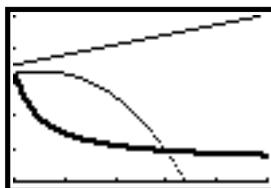


Figure 4. Graphs of three models for the period 1995 onwards similar to that produced by C2.

Students' lack of understanding of the relationship of the models they had produced to the context they were modelling was evident particularly with those students who produced only a qualitative description as their solution to this part of the task. Often no mention or use was made of the linear model, which assumes no new treatments.

Bb14: Model b [quadratic] estimates that between '95 to '98 the number of deaths decreased quickly at first then the rate decreased slowly - (shallowed out), as represented by the parabola.

Model c [hyperbola] does not show the number of deaths from '95–98, but rather the decline of deaths from '98 onwards to 3 deaths per year in the long term.

Students also had difficulty discerning that the assumptions being asked for were those associated with the particular formulation of this aspect of the task. Often they merely reiterated the assumptions they had made with the models they had set up previously. Others realised that this part of the task involved further assumptions related to when the therapy began to take effect but focussed on these minor issues.

Bb8: The result would be understated because the number of deaths were not considered during 1994–1995. The drugs were only introduced during that time frame, but it was not stated when (e.g. beginning of 1994 or end of 1994 or end of 1995).

The vast majority of the students (35) believed that the reduction in the death rate was an indicator of lives saved.

Bb10: The reduction would be an overall saving of lives because less people are dying of the disease. Although it may not be as effective at the start, there will be an overall improvement in health.

Three students spoke of it as lives saved plus the effect of prevention, introducing factors extraneous to their models.

C2: The reduction is caused by lives saved, but there is also, more and more people heard of the disease, and start having prevention of it.

The two best explanations saw the reduction in the death rate as an indicator of prolonged life. A third student saw it as a reduction in the progression to AIDS.

- Ba15: The reduction isn't lives saved because you can cure AIDS; but it is prolonged lives.
- Ba3: The reduction is the amount of people progressing from HIV to AIDS and AIDS related deaths per year compared to how many deaths there would have been had the new therapies not been introduced. It is a *reduction* of deaths only which means a reduction of people progressing to AID's and dying from it.

Modelling Australian AIDS data

The Australian data consisted of tabulated new AIDS cases diagnosed by quarter from the end of 1982, when the disease was first discovered, until the end of 1988. A large number of models was used by researchers to model AIDS incidence in Australia in the early years of the epidemic. All of these were based upon the exponential model. If n represents the number of new AIDS cases diagnosed in quarter t , where $t = 1$ corresponds to the last quarter of 1982, three of the best fitting of these models (Wilson, 1989) that are of a degree of simplicity accessible by students in Mathematical Methods courses were:

$$\ln(n) = 2.7 + 0.083 t, \text{ for } t \geq 20$$

$$\ln(y) = -0.446 + 0.421 t - 0.0086 t^2, \text{ for } t \geq 20$$

$$\ln(y) = -1.64 + 2.035 \ln(t), \text{ for } t \geq 20$$

Among other things, students were asked to draw all three functions and their gradient functions (Figure 5) for $t = 20$ until 1993 and to describe what the three functions predicted about the number of AIDS cases in the long term (going well past 1993) and what each gradient function predicted about the rate of change in the number of AIDS cases in the same time period.

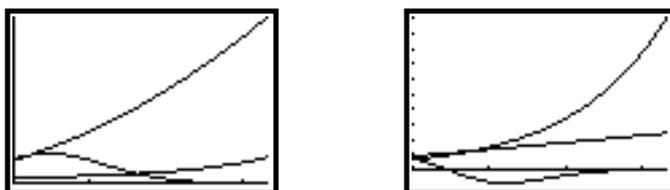


Figure 5. Graphs of the three function models for Australian AIDS data and their gradient functions.

Students produced sketches of the graphs with the help of the graphing facility on their calculators or used graphing software such as *Graphmatica*. These descriptions ranged from being cursory (e.g. Bb6) to quite detailed (e.g. C12) but examples of the latter were in the minority.

Bb6: $y_3(t)$ predicts that the rate of change is constantly increasing.

$y_2(t)$ predicts that in the future the number of AIDS will continue to rise rapidly and shows no sign of levelling out or decreasing in the future.

$y_1(t)$ predicts that the number of cases will steadily decrease until it has levelled off to which it will infinitely increase along the t-axis.

C12: The gradient function (1) $\left[\frac{dy}{dx} = 0.83e^{2.7+0.083t} \right]$ is like the original graph and

shows it is increasing in a positive direction. There is a steep rate of change at 6.5 on the 20th quarter and by the 45th quarter rising to 51.7. The gradient function of graph number (3) was also similar to the above. $\frac{dy}{dx} = 0.39t^{1.04}$ also increases, however at a slower and steadier rate only increasing to 20.3 in the 45th quarter from 8.8 in the 20th quarter. These two graphs will also intersect at the 28th quarter reaching 12.5.

The gradient function of no. (2) $\left[\frac{dy}{dx} = -(.421 - .0172t)e^{-.446+.421t+.0086t^2} \right]$

decreases unlike the other two graphs, going into the negatives where it hits the graph at the 25th quarter. This indicates the x-coordinate of the T.P. This indicates that rate of change is dropping, starting at 7.2 at the 20th quarter and ending at -1 in the 45th quarter.

General observations

The widespread use of CAS calculators by students to support their generation of models for the task has ensured the production a wide variety of models relatively free of mechanical errors related to algebraic manipulation. It has also meant that there were few aspects of the task that any students did not attempt or any gave a very cursory answer except for interpretative questions involving reflection on models within the context. Even within such a fairly structured task, students have been able to adapt standard methods and to choose from a variety of models at their disposal. However, generation of models is only one aspect of mathematical modelling. With the accuracy of this process being facilitated by the CAS calculator environment, there is more time for careful consideration of the conditions of use of various mathematical techniques and reflection on the underlying assumptions and limitations of these models especially in relationship to the task context being modelled.

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Enriching student learning of pattern and algebra by enriching teachers' understanding*

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This paper reports on a school-based action research project focussing on developing primary teachers' knowledge and understanding of pattern and algebra, and the resulting outcomes for students. The project was instituted and constructed by the staff of St Monica's Primary School in the ACT, with academic support from the University of Canberra. The results suggest that the depth of teachers' own knowledge and understanding is critical in changing the way we look at mathematics, and hence the pedagogy of the classroom.

Developing the research focus

A few years ago, St Monica's Primary School in the Australian Capital Territory changed from a structure in which students were ability-grouped for mathematics and followed a commercial textbook series, to one in which they remained in their heterogeneous class groups and in which the class teacher was responsible for developing an inclusive mathematics learning program. The school staff felt that this had been a positive change, with students more motivated, but wanted to take the next step in creating a challenging and enriching mathematics for all. Accordingly they decided that mathematics would be a professional development focus for 2002. They already participated in numeracy projects such as *Count Me In Too* (NSW Department of Education and Training, 1998), but wanted to broaden the focus of their professional development to other areas of mathematics. The Pattern and Algebra action research project grew out of this desire to develop a richer mathematical and problem solving environment for students.

The project was structured to provide a balance of expert input, relevant professional reading, teacher reflection and collaboration with colleagues in the classroom. Teachers monitored their and their students' learning by reflecting and reporting on their observations.

* This paper has been subject to peer review.

Initial input

As the academic consultant to the project, I conducted a one-day seminar involving all staff prior to the commencement of the school year. At the seminar I outlined my thoughts on the directions of mathematics education over the past few decades, and where I thought we had progressed or otherwise.

The presentation included a brief overview of curriculum changes such as the so-called 'new maths', the movement to a problem-solving focus in mathematics, and the development of curriculum statements (AEC, 1991). Staff discussion supported the view that while each of these developments contained much that was worthwhile and had made some important differences to school mathematics, they had ultimately not led to the changes in classroom practice that had perhaps been envisaged (Lovitt, 2000).

We then looked at recent research such as that conducted by Ma (1999) and Askew et al. (1997), which have focussed attention on the need for teachers of mathematics at all levels to have a deep and connected knowledge of mathematical ideas.

A staff focus group was formed, who then discussed the implications of the presentation and the particular needs of the school. A consideration of the limited impact of some of the historical movements in mathematics teaching in Australia, particularly in the context of this organisational and philosophical change in the school's own mathematics program, led the group to a decision that lasting change in teacher pedagogy and student learning was dependent, in large measure, on the staff's own confidence in, and beliefs about, mathematical knowledge. In discussion with me as an external consultant, the group chose to focus on visualisation, particularly in algebra, as the focus of their professional learning.

Developing teachers' knowledge and understanding

Throughout term 2 the focus group met every two to three weeks for approximately two hours after school. These sessions focussed on developing deeper mathematical understanding through visualisation, and on the use of hands-on materials to illustrate significant mathematical concepts.

At the initial session we focussed on the three underpinning conceptual ideas of arithmetic and algebra: the commutative property of addition and multiplication, the associative property of addition and multiplication, and the distributive property of addition over multiplication. We talked about how these ideas underpinned many of the mental computation strategies developed in materials such as *Count Me In Too*, as well as the traditional written algorithms for addition and multiplication. We also looked at how algebraic processes such as $3a + 4a = 7a$ or $(x + 5)(x + 7) = x^2 + 12x + 35$ were really just extensions of these mental and written processes.

In particular we looked at visual representations of these three underpinning ideas using both concrete materials such as counters or MAB blocks, and computer-generated representations using *PowerPoint* or virtual manipulatives (Utah State University, 2002). Although representations such as these are not new, the opportunity to use concrete materials, and the dynamic nature of the computer images, helped

teachers to make better connections between the specific processes of arithmetic and the more general abstract concepts of algebra.

The transition from arithmetic to algebra depends, in large part, on the capacity to see the big picture holistically. As noted in Thornton (2001) visual thinking activates the holistic and structural thinking processes characteristic of right brain thinking (Table 1).

Table 1. Left brain and right brain thinking (Thornton, 2001).

Left brain thinking	Right brain thinking
Verbal	Visual-spatial
Analytical	Synthetic
Symbolic	Concrete-pictorial
Logical	Intuitive
Sequential	Multiple processing
Linear	Gestalt, holistic
Conceptual similarity	Structural similarity

In later sessions we expanded the use of visual representations to an investigation of divisibility tests (Bennett and Nelson 2002) and the generalisation of arithmetic patterns such as

$$\left. \begin{array}{l} 1 \times 3 + 1 = 4 = 2^2 \\ 2 \times 4 + 1 = 9 = 3^2 \\ 3 \times 5 + 1 = 16 = 4^2 \end{array} \right\} \text{(Thornton, 2001).}$$

While the focus of these sessions was on developing deeper mathematical understanding of arithmetic and algebraic concepts, the teachers discussed the potential implications for their own classrooms. At the last of these sessions, we discussed some classroom activities that would help students to develop ideas of pattern and algebra in which visualisation played a central role. This formed the basis of the teachers' action learning with colleagues.

The action learning process

Input at these focus groups sessions was supported by a collegial dialogue. The teachers formed research partnerships consisting of two teachers who taught at adjacent grade levels. In this way they could gain a sense of where students were coming from and what might be possible. Within these research teams they planned lessons based around the work in pattern and algebra, and arranged to observe and comment upon each other's lessons. The group then met to discuss their lessons and reflect on the process and what they had learned.

This action learning was grounded in four underlying principles (Shulman 1990):

- Teacher volunteers (the learners) are expected to be actively involved in their own learning and formulating relevant questions to explore.

- Teacher volunteers (the learners) are encouraged to work collaboratively to support and challenge each other.
- Teacher volunteers (the learners) reflect upon what, how and why they are learning.
- Teacher volunteers (the learners) are nurtured within a community of learners in their professional field.

Results

Kindergarten class (Carmel)

The Kindergarten class focussed on the outcome, 'Recognises, describes, creates and continues repeating patterns and number patterns that increase or decrease' (NSW Board of Studies, 2002, PAES 1.1).

They began their unit of work using a variety of concrete materials to 'make a pattern'. The teacher used this language deliberately as she wanted to gauge the children's knowledge of the term. Only three children displayed a repeating pattern. Most created constructions which they named and referred to as patterns, but which were, in reality, random collections of objects.

Carmel then provided input using the children to make and identify patterns. They used examples such as boy/girl and hair colour, using the term pattern and voicing this out loud: 'The pattern is boy, girl, boy, girl'. The children then made patterns using felt pieces and pattern blocks (Figure 1) and predicted the next element in the pattern, developing the idea of a repeating pattern using a variety of materials.



Figure 1. A pattern made by Kindergarten children.

Carmel also created a *PowerPoint* presentation in which children were asked to predict the next picture to appear on the screen, and showed this to the class using a data projector. It generated great excitement, particularly when the pattern was unexpected.

Year 2 class (Helen)

The Year 2 class used a clock as the vehicle for exploring patterns. The outcomes focussed on were, 'Extends a variety of number patterns and supplies missing number elements and builds number relationships' and 'Compares and orders the duration of events and reads clocks on the hour and half-hour' (NSW Board of Studies, 2002, DAS 1.1, MS 1.5).

After exploring ideas such as hours, minutes and seconds on an analogue clock, and several group activities looking at representations of clocks, children each made their own analogue clock using a paper plate, brenex squares and paper fasteners. They also made group clocks using popsticks (Figure 2). They then explored and described some of the number and object patterns in the clocks. They observed, for example, that multiples of 5 were used on the clock face, and that the pattern of lines around the clock was a repetition of four small lines and one large one every five minutes. A particularly valuable learning experience was that putting five lines between each set of five minutes resulted in too many minutes in each hour.



Figure 2. A clock made from popsticks.

Year 3/4 class (Alison)

The focus of the Year 3 and 4 class was 'Generates, describes and records number patterns using a variety of strategies and completes simple number sentences by calculating missing values' (NSW Board of Studies, 2002, PAS 2.1).

Over the course of the first three lessons the children recorded multiples on a large 100 grid. Each counting pattern from 1 to 10 was recorded using different coloured dots on the grid. The children then identified, explored and recorded as many patterns as they could find.

The class then discussed what had been discovered about multiples and factors using the 100 grid as a reference. Each child chose a number between 1–100, physically created a tree from coloured paper and illustrated the factors for their number on the leaves of their factor trees.

Finally the children created a pattern detective display as the end result of investigating the patterns of multiples. The children coloured in blank and numbered 100 squares and then played a guessing game in which they had to guess which pattern was being shown.

Of particular interest was a class project to produce a large 100-strip, showing numbers to 100 horizontally, with all factors of each number highlighted in the rows beneath (Figure 3). This display enabled children to investigate questions such as the number of different factors each number possessed, which numbers had only two factors or which numbers had an even number of factors. Even though the children had not yet encountered ideas such as square or prime numbers, the display laid a strong visual foundation for further work in this area.

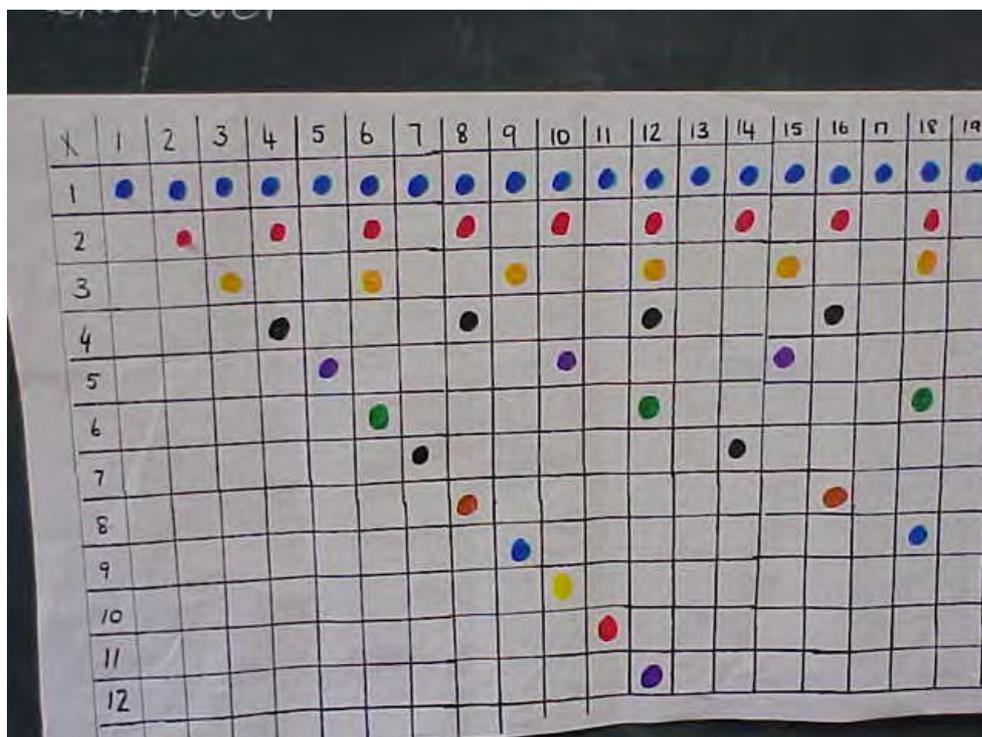


Figure 3. A hundred-strip.

Year 6 class (Anthony)

The Year 6 class focussed on the outcome 'Records, analyses and generalises geometric and number patterns that involve one operation using tables, graphs and words' (NSW Board of Studies, 2002, PAS 3.1a), starting with some traditional matchstick pattern activities (Thornton, 2001), and then looking at patterns related to square numbers. The students were surprised that the graphs of the two relationships looked different.

The students were then given an open problem to explore patterns of overlapping squares, made with squared paper and concrete materials such as multilink cubes. Looking at the overlaps between adjacent squares enabled the students to investigate and make conjectures about ideas such as the differences between square numbers (Figure 4). The students also looked at Fibonacci numbers, and conducted Internet research to find out about the occurrence of Fibonacci numbers in nature.

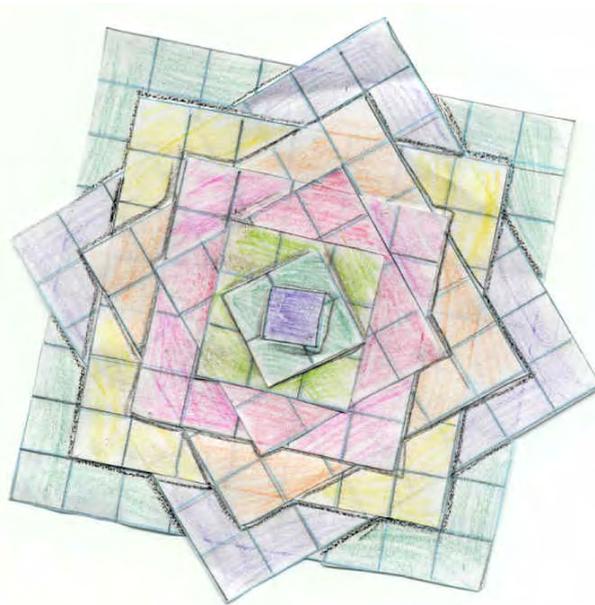


Figure 4. A series of overlapping squares.

Conclusions

The original research question posed by the teachers was 'Does improving teachers' understandings of concepts in the area of numeracy improve teacher pedagogy and lead to improved student outcomes?'. While there are no specific quantitative measures of student outcomes, teacher observations would certainly indicate that the children learned a great deal through their teachers' focus on pattern and algebra.

Visually representing addition and multiplication counting patterns on the 100s chart allowed all the children to see and discover new patterns. It also opened up opportunities to explore and introduce new terms such as multiples and factors. We also continued to explore new ways of visually representing counting patterns

on 100s charts. As a result of this work, some children have continued to explore other patterns they see in number and it has also given confidence to those children who usually would not have a go. (Alison, Year 3/4 teacher)

The children showed very high levels of interest and motivation. Looking at maths in this way is new for them and I was very surprised and pleased at how quickly they were able to generalise the patterns. Their ability to use algebraic terminology was terrific. (Anthony, Year 6 teacher)

The teachers in the project reported that the action learning process had been valuable for their own learning and pedagogy. Perhaps the most important aspect of this was that teachers saw themselves as learning alongside the children. They developed the confidence to allow children to explore patterns, even if they did not know the answers themselves.

Planning and discussing my learning with a colleague was fun and kept me focussed. (Helen, Year 2 teacher)

The opportunity and time to plan and implement this project has been very beneficial. I have learnt a great deal, particularly through the input... on the visual elements of patterning and algebra. (Carmel, Kindergarten teacher)

This experience has really helped me understand the real importance of visualising maths concepts I would not have previously thought of or had the confidence to try. This process has certainly made me rethink how I approach maths teaching and learning in the future. (Alison, Year 3/4 teacher).

As a researcher, the project has highlighted for me the importance of teacher ownership. This is nothing new, but it is perhaps the key element in all effective professional development programs. During the project I could not help wondering whether or not anything worthwhile was actually happening, as there were considerable periods of time when the teachers and I had little contact. I did not realise the extent to which the teachers involved were building on the ideas we had talked about, and implementing them in their classroom. The teachers had both the space and the support from colleagues to pursue outcomes they considered important.

To what extent is the process the teachers undertook at St Monica's transferable to other teachers and other settings? Will the project result in long term benefits for the teachers and students involved? These questions remain to be answered, but we hope to continue the action learning next year with another group of teachers in another content area.

Acknowledgments

The research and professional development was supported by a Teaching Futures grant provided by the Catholic Education Office of the Archdiocese of Canberra and Goulburn.

The author wishes to thank the teachers involved in the project and the administrative staff of St. Monica's Primary School for their support.

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New technologies, new tools, new thinking

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This paper outlines the background experiences and thinking of a practising teacher working to formalise her thinking toward a doctoral dissertation. Many years of watching young learners work with mathematics, as presented by many different teachers, has resulted in an investigation into the mathematics and numeracy required in the twenty-first century within the context of the new technologies; and so, with new physical tools which provide new thinking spaces, the doctoral study will investigate ways children can use graphic organisers to think about mathematics in the way they will need to if they are to be numerate.

This paper is the account of the thinking of a practising teacher as it develops into a doctoral study. The thinking in this paper developed through twenty-seven years of working with children in mathematics.

Where did this thinking come from?

Teaching mathematics, watching children learn (or not learn) mathematics, and observing mathematics being taught by a wide range of teachers across the full pre-school to Year 7 range, has provided much information and pedagogy for reflection. Observing mathematics being taught to children as a series of unrelated learnings that only exist for the purpose of accurate computation and being a part of three new syllabus developments over the period of my teaching career has given rise to the considerations of this paper to be followed up in my investigations for my doctoral study.

Early personal experiences that created an interest in exploring mathematics education further were:

- my own experiences as a low level mathematics learner in high school, having spent numerous hours in rote completion of sums from mathematics textbooks; and
- the fact that I came to understand multiplication and its characteristics in my second year of Teacher's College during a session on how to use Cuisenaire in classrooms.

Professional experiences that have influenced my thinking are:

- the many times that children have told me that they hate mathematics, referring to computation, and that mathematics has nothing to do with measurement, space, time, money, and chance and data, or making meaning;
- syllabus documents representing mathematics as different strands of study, often unrelated;
- teachers who have frequently told me that mathematics is not connected to other areas of the curriculum and could not be integrated with other learning;
- the constant insistence of teachers in the eighties and nineties for a school program that instructs them as to the mathematics program to be taught at their level — often a content-based school program was expected by them, listing content to be ‘given’ to the students; and
- observing the sheer frustration from many (not all) teachers when provided with a school program that presents a developmental sequence that needs to be matched with the learners in their class.

Ausubel (1968) suggests that all learning is influenced by prior experience, and my personal and professional interest in the teaching of mathematics is surely so. The list above are influences on my thinking and in no way a criticism of others who understand mathematics education to be something different. It is well documented that our challenges arise from the massive gulfs that originate from a comparison between our own thinking and that thinking from others around us.

Therefore, the contents of this paper and my presentation seek to provide the substance of substantive conversation amongst all teachers whose professional responsibility it is to engage learners with mathematics and numeracy.

The immediate past

The outcome to be achieved is a way by which children understand that mathematics lives far beyond the algorithm and computational competency from an early age. This involves finding a way for children to build a conceptual understanding network that is useable and flexible. Children build into the network, knowledge, patterns and relationships that change over time. Building is a process over time, from many perspectives and influenced by experience.

The idea that learners can be visual or auditory or kinaesthetic or some combination of these at any one time is one regularly proffered for our consideration in schools. The challenge of this study is one of manipulating ideas and knowledge, visually and kinaesthetically, to make the connections, to see the patterns, to recategorise smaller ideas to form a bigger picture view.

The eighties’ Queensland Mathematics syllabus adopted the terminology of ‘visual representation’ as a process to be taught in mathematics. Diezmann’s (1996, 1997) many papers on visual technique emanate from the ‘draw a diagram’ approach presented in the documents of the time.

Insight suggested that visual representation was to be used as a tool for solving problems, not for and of its own value. Some interpretations related it to learning about

graphs; others related it to the ‘symbol, concrete, pictorial’ triad representation (for example, ‘show me three blocks’) and left it at that.

My interest in visual representation: is it the same as visual thinking, visual imagery, or visualisation? Did the syllabuses really limit visual representation to representation only or were we meant to have deeper insight into it as visual thinking? In an era of technologies, visual thinking in its fullest meaning must be what we mean.

The study

In refining a topic of study, the major issues of interest came to be:

- the mathematics with which children engage;
- the mathematics-numeracy dimension; and
- the use of visual thinking in mathematics.

The era of the study dictates a technologies focus that of itself dictates the visual thinking aspect and the increased priority for mathematics education and numerate persons.

The context of this study is one of educational reform worldwide and hence, cognisance of an atmosphere of professional challenge underwrites this study.

The investigation

My intent was to investigate the use of visual thinking on the computer by students in Years 1–3; to construct and edit graphic organisers over time to develop competence and productivity in shaping and selecting a new and deeper knowledge of numeracy.

Purpose

The purpose of this study is to determine the uses young children make of visual thinking in selecting and shaping their numeracy knowledge and understandings. The study focusses on the open-ended use of a visual thinking tool by children. Once skilled with the functionalities within ‘Kidspiration’, students use it to enhance thinking about their numeracy learning. This representation of their thinking, knowledge, concepts and linkage, forms the basis of substantive discussion with peers and teachers.

This study identifies the uses students make of visual thinking in all areas of numeracy, that is, number, operations, measurement, chance and data and space. This is consistent with the current definitions of numeracy that have widened the boundaries of numeracy beyond number and are inclusive of processes of selection of information and data and critical analysis of uses and application of data.

This study presents visual thinking as the tool to develop the abilities required by the numerate person, to make the linkages, to develop the depths of thinking and bring them to the surface for use and communication to others. Visual thinking is the means by which the private nature of learning and thinking is facilitated and made public. It is a tool by which the individuality of our thinking is constructed.

Visual thinking uses graphic organisers to provide different ways to think: to order and organise ideas and knowledge, and to develop new knowledge. 'Kidspiration' for younger children and 'Inspiration' for older children is software, which integrates enabling choice or construction of various visual structures for 'selecting' and 'shaping' knowledge.

'Kidspiration' is developed for children aged four to ten years. It is a tool for 'brainstorm[ing] with pictures and words, organis[ing] and categoris[ing] information, creat[ing] stories, and explor[ing] new ideas' ('Inspiration', 2002).

What is the view of mathematics and numeracy for this study?

Given the complexity and changed characteristics of the literacy and numeracy practices being dictated by the information communication technologies, participants in social cultural practices will need to interact critically with linguistic, numerical and graphic information (Department of Tasmania, 2002, p. 40). Communications engage the participant in interpreting and generating these three forms of representation interwoven in one text.

Beyond the discussion of literacy and numeracy are the discussions to differentiate mathematics from numeracy. For this study, mathematics and numeracy are portrayed as the study of bigger ideas, the process of conceptualisation of processes and concepts into systems that are flexible and can be acted upon to respond to various scenarios. Numeracy is differentiated by its relationship with everyday scenarios. Generally, in the early childhood context of this study, differentiation serves little purpose. Mathematics and numeracy will be viewed interchangeably.

Common to all categories of literature in relation to this topic is the meaning-making purpose of mathematics and numeracy. Some aspects of mathematics are for the beauty of the proof or the theorem in and of itself. Numeracy, though, is for the purpose of meaning-making of the complex systems in which we participate everyday.

This study rates as important that mathematics and numeracy are seen as far more than computational competency and manipulation of symbols. The relationships and patterns, concepts and processes must be explicit and available to the learner who is developing a big picture of mathematics, the outcome of mathematical learning to be valued. To serve the 'new look' of mathematics and numeracy, different ways of thinking and different contexts in which to think are highly relevant as is the corresponding new knowledge that eventuates.

Willis (2002) concluded that our focus on the procedural knowledge, the small mathematical ideas or concepts, 'withholds from children the very connections and understandings that will sustain them in the long term' (p. 7). Clarke, Cheeseman, Sullivan and Clarke (2000) designed a framework focussed on this point, looking at the connections and how children are helped to make them form the bigger ideas. Grouws and Cebulla (2002) referred to these big mathematical concepts as 'important mathematical ideas' (p. 1). Their study of the research findings also concluded that

development of conceptual understanding early enables effective procedural knowledge usage later.

Several studies (Clarke et al., 2000; Grouws & Cebulla, 2002; Willis, 2002) conclude that the need is for robust learning and long term performance (Willis, 2002, p. 8). The study by Clarke et al. (2000) reflected on the change to focus on concepts, links and connections and ‘pull[ing] it all together’. Perry (2000) described this as facilitating success for learners by ‘linking their learning experiences to their context’ (p. 22) for substantive learning.

Warren and English (2000) focussed on the knowledge of mathematical structure with sets of mathematical objects that are related in their study of the impact of classroom practices on deriving this structure. They argued that emphasis on computational procedures limits understanding of structure. In a study of ninety-four children aged ten to twelve years of age, they explored the ability to understand structure and relationships from problem situations. They found evidence that students leaving primary school have no notion of mathematical structure and failed to abstract relationships and principles from algebra in tasks in the study. They concluded that a balance between computation and searching for implicit patterns is required along with a need to explore non-relationships.

Lesh (2000) discussed the need for students to construct their own conceptual frameworks. In indicating that schooling is now about making sense of complex systems, Lesh stressed that ‘maths involves more than simply manipulating mathematical symbols’ (p. 193). He suggests that children must have access to powerful conceptual amplifiers in the form of language and diagrams to express concepts and links in powerful ways. These tools must be ‘sharable, reuseable and modifiable’ (p. 194). Lesh suggests that representational fluency is at the heart of what it means to understand most mathematical constructs. He links technology with the sophistication of more powerful systems and these new systems and related behaviours link to the development of new technologies.

Earlier work by Johnson (1987), Presmeg (1986) and Davis (1984) suggested students were not gaining an intuitive ‘gut feeling’ for mathematics (p. 177). Students were missing the point of mathematics, learning theorems by heart for the purpose of passing tests or for the inner mathematical beauty. The ‘new look’ mathematics and numeracy developing from the work of Clarke et al. (2000) and Willis (2000) brings to life the work of earlier research, all of which cites the need for the view of mathematics and numeracy as a mass of concepts and relationships in a bigger picture. Mathematics has organisers or themes that bring coherency to the picture. The ‘new look’ does not encompass the picture of mathematics and numeracy as a huge set of independent facts and knowledge that is to be viewed from one perspective only.

The ‘new look’ mathematics and numeracy defined

These studies identified the ‘new look’ mathematics as having a dual perspective. The ‘new look’ mathematics and numeracy is a mouldable mass of concepts and processes in which embedded relationships are selected and shaped in a meaning making process for the learner to make sense of complex situations and to participate equitably in life

practices. Computational competency has a role within mathematics and numeracy of the twenty-first century and can prejudice the bigger view of mathematics and its structures required for flexible effective mathematical thinking.

The 'new look' raises questions for the types of classroom practices that support the wider character of maths and students generating a structure that increases with complexity from within maths and within the experiences of each individual and the nature of their personal experience and participation. Teachers are faced with the challenge of building an environment in which students bring procedural and conceptual understanding into a schema that they modify and rebuild to match personal everyday challenges. Mathematics and numeracy are dynamic through the relationships of concepts and concepts, and concepts and relationships.

How is this concept related to this concept? What are the similarities? How are they different? What is the link? How consistent is the link across scenarios?

How is this concept embedded in this relationship? How does this concept not work in the relationship represented here?

Visual thinking

The literature in visual thinking, visual imagery and visualisation suggests that students actively engage with the construction of representations. The construction generates the formation of structure, recognition of relationships and the integrative nature of mathematical learning. Research literature suggests that teaching actively promote learning of visualisation and variety of representation for it to have impact.

Van Hiele carried out much early influential work in the field of visualisation and mathematics (in Yelland & Masters, 1997). '[S]uccessful students are not those who learn isolated facts but those who are able to construct networks of relationships that link geometric concepts and processes' (p. 84). Identification of systematic patterns scaffold movement toward abstractions.

The hypothesis for the current study is that visualisation in the form of representations using graphic organisers will provide the 'sharable, reuseable and modifiable' tool in which children can think about the bigger concepts of mathematics, over time, building the connections and the flexibility in thought and usage of the concepts and connections; that which Noss and Hoyle (1996) call webbing.

Visual thinking in mathematics and numeracy

From studies with the use of graphic organisers as visual images, over time, it is to be concluded that learners develop flexible use of an effective tool, to be used or discarded dependant on the level of interaction with the conceptual understanding, comfortable or novel. The student learns to select currently appropriate information, modify and build it consistent with the requirements of the new situation. Visual schemata are dynamic and generation is the process of identifying appropriate elements through recognition of patterns and order. Generation is the deep thinking using deep understanding for a new context: new knowledge results.

The studies would suggest that no sequential development of visualisation is evident, more a differential usage to match the context, its content and complexity. Usage needs to be developed by the teacher, with students understanding that it is the thinking space and time of building the graphic organiser that contributes to learning, and that the learning is concerned with the bigger picture of mathematics.

Thus, there is general support that visual thinking has learning benefits in deeper thinking and understanding when embedded in the working processes of the learner over time. The learner needs time and support to generate their own visualisations. The learner needs to understand that visualisation is characterised by flexibility and individuality. Visual representations are dynamic and personal thinking spaces.

Engaging with visualisation over time in an environment in which processes are enhanced through explicit guidance and respect for the mouldable mass of mathematical information, develops results in the development of thinking mathematically. Children will access an impressively powerful conceptual tool that uses language, diagrams and symbolic description (Lesh, 2000, p. 190) to:

- focus perception;
- consider the unexpected;
- seek connections, metaphors and analogies; and
- invent innovative solutions (Department of Education, Tasmania, 2002, p. 15)

as well as generalise.

Work that has identified the uses made by various groups of learners suggests that, for all learners, the ability to think visually and bring order to mathematical understandings and relationships could support deeper understanding — to all learners, both high and low achievers. Students can be taught to use a tool which enables them to decide on necessary information, focus on process — not step by step — attend to outcomes and edit information and structure.

Conclusion

Placing this view of mathematics and numeracy and mathematical pedagogy in the context of technologies, Noss and Hoyle (1996) suggest that our focus ‘is on what the computer makes possible for mathematical meaning making’. Accepting that ‘big picture mathematics and numeracy’ is our ‘new look’, integrative and generative approaches need to be investigated for building into classroom learning.

A disposition to engage with processes of collecting, selecting, choosing, assessing, shaping, using and applying are as much a part of being numerate in the twenty-first century as they are of being a mathematician.

‘[F]ocussing on sense making in mathematical reasoning and using facts, properties and relationships to make and test conjectures and follow and develop logical arguments’ (Yelland, 2000, p. 49; Clarke et al., 2000; Bobis, 2000) requires teaching of mathematics and numeracy. Teaching needs to provide opportunities for children to work for a greater time at greater depth with the big picture in a thinking space, both flexible and dynamic.

This study approaches mathematics and numeracy from a non-linear, conceptual approach that focusses on deeper understanding through a view of mathematics and numeracy as a mass of concepts and processes as objects, and relationships that can be ordered and patterned differently for different purposes. Children need opportunity to find the patterns, make the relationships and generate schemata that they use and modify to meet their needs.

Analysis of data 'lay[s] bare the internal differences that are hidden by external similarities' (Cole et al., 1972, p. 63). Analysis seeks to reveal the dynamics between the internal behaviours of young students:

- between teaching and learning;
- between structures with which they are confident;
- over time; and
- the resultant productivity.

Analysis brings to the surface the 'idiosyncratic ways' (Luke, de Castell & Luke, 1989) and the student centredness (Queensland School Curriculum Council, 2002) of thinking. Students choose to select and shape, deconstruct and reconstitute their learnings in their own personal ways.

Given the revelations of the analysis, this study seeks to place the understandings of visualisation from the research in the bigger context of the mathematical mass, concepts and processes and relationships, using graphic organisers in the context of technologies to provide a generative and integrative environment for children.

Whatever the findings in an academic sense, being on this journey with young learners over the months to come will provide insight into the thinking and productivity of young learners in their use of a classroom tool, 'Kidspiration' when embedded in classroom practice.

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Re-learning in a CAS-aided classroom — a teacher reflects*

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Throughout 2001–2002, students in the author’s Year 11 and 12 Mathematical Methods classes took part in a pilot course in functions and calculus in which ubiquitous access to a Computer Algebra System (CAS) calculator was assumed, both as a learning tool and in all high stakes assessment tasks. In this paper, the author reflects on his experience and those of his students as they adapted to this new way of teaching and learning.

The CAS-CAT Project

A research project in Victoria is investigating how mathematics can be taught, learnt and assessed when students are permitted access to a CAS in examinations. The University of Melbourne is conducting a three-year research study (2000–2002) entitled *Computer Algebra Systems in Schools — Curriculum, Assessment and Teaching (CAS-CAT) Project*. This project aims to explore the feasibility of offering new mathematics subjects that incorporate the extensive use of CAS, both in the classroom and in formal assessment.

Mathematical Methods (CAS), which involves the study of functions, calculus and probability distributions, is a new subject being piloted in Year 11 and 12. Students undertaking this subject are required to sit two CAS-permitted examinations. Further content and administrative details can be accessed from the Victorian Curriculum and Assessment Authority (VCAA) website (<http://www.vcaa.vic.edu.au/vce/studies/MATHS/caspilot.htm>).

This is not a research paper reporting on the formal data collected in the CAS-CAT project — the results and analysis of this data will be published elsewhere (see for example, a list of published papers at the CAS-CAT project website <http://www.edfac.unimelb.edu.au/DSME/CAS-CAT>). In this paper, the author reflects on his experience as a classroom teacher involved with the project.

* This paper has been subject to peer review.

What was different?

Students in the project had ubiquitous access to a CAS calculator with a symbol manipulator, and therefore many of the symbolic procedural tasks we had previously expected them to master were trivialised. In this knowledge, the University and the VCAA contributed many new tasks that attempted to promote new ways of working and thinking. These tasks started students thinking about abstraction earlier than we had previously. As teachers, managing this was sometimes difficult, since text books tend to favour a bottom-up approach: many examples, then student exercises graded up to worded or abstract questions. Some of the new materials were more ambitious, introducing parameters and general relationships early, as well as the introduction of general solving algorithms. We felt challenged to work in a new way, spending more time on more substantial investigative questions. Many of our students were used to the ‘textbook’ style of working, and initially found this focus on fewer, deeper questions to be somewhat frustrating.

In the early weeks of the project, we tried to give Year 11 students experience of the use and meaning of functional parameters, but seriously overestimated how quickly students would pick this up. Here is an example:

Consider the family of functions $\{f_n\}$, $n \in \mathbb{N}$, defined by the rule $f_n(x) = x^n e^x$ over domain \mathbb{R} .

- Draw the graphs of f_n for $n = 0, 1, 2, 3, 4, 5$ and find exact coordinates of the stationary points in each case.
- Find a general expression, in terms of n , for the coordinates of the stationary points of f_n , and the range of f_n .
- Find an expression for the anti-derivative of f_1 .
- Find a general expression for the anti-derivative of f_n in terms of an anti-derivative of f_{n-1} .

Based on our experience thus far, we will need to improve how we teach with more abstract questions about parameters, and ‘show that’ questions. For example, there is some evidence that students do not always recognise the need to use the CAS in such questions (Asp & Flynn, 2002). In the context of the pilot project, some of these big questions were inadequately treated, due in part to teacher inexperience and a desire to keep our course on track. In the future, we will plan for more longer analysis tasks for students each semester where they are able to spend more time on these parameter explorations.

By hand, by head, by CAS

One of the major themes for us throughout the project was maintaining a balance between students ‘by hand/head’ and ‘by CAS’ work. It would be fair to say that we often erred on the side of the former, informed by a belief that new most mathematical procedures are best introduced in an ‘organic way’. Take the quotient rule for differentiation as an example. When instructing students about this, we made use of the following sequence.

1. Set the context for the procedure.
2. Illustrate the link between it and the product rule.
3. Show some examples in which it may be helpful or more efficient and/or reliable than other methods (for example the product rule).
4. Ask students to perform the procedure in a simple cases.
5. Introduce a more complicated example, and ask students to compare their by-hand answer with the CAS result.

We expected all students to be able to perform each analytic method for straightforward cases. Apart from the ‘sense-making’ objective, we believe students needed some basis for discerning whether it would be more efficient and/or reliable to complete the procedure by hand.

In summary, for each procedural algorithm being studied, we insisted that students be able to:

- perform the algorithm by hand in simple cases;
- recognise when the algorithm was necessary;
- decide when it is quicker and equally reliable to perform the algorithm by hand.

Of course this takes time, and not all students respond equally to this expectation. In fact, there were some students with strong analytic skills who were initially reluctant to use the calculator for symbolic manipulations, out of a belief that their skills would atrophy. These students eventually became convinced that some operations were either too difficult or too long to complete by hand. Interestingly, these students were also often the best at discerning when to pick up and when to put down the calculator.

By contrast, a small number of students became over-reliant on the CAS calculator, and often chose to use it when it would have been slower than completing it by hand (and in some cases by head!). It is possible that their relative lack of confidence in by-hand algebra contributed to this preference. These students were also often less able to interpret the CAS output, or to make sense of the error messages produced.

A variety of methods

We found the presence of symbolic, numeric, and graphic representations combined in the one calculator to be very useful, particularly as an instructional tool. It was particularly helpful to use the overhead display of the calculator to help us to:

- train students about appropriate syntax;
- model the process of switching between representations (e.g. graph, table, symbols);
- permit students to demonstrate alternative methods, or show interesting results;
- project display onto whiteboard, and then annotate output as necessary.

In common with the non-CAS graphics calculator, the overhead display allowed us as teachers to model ‘ways of working’ so that students could develop greater efficiency

with the CAS. However it was the questions and conjectures that we could make which was most valuable for learning, as we sometimes chose to use the symbolic features of the CAS, rather than always retreat to the whiteboard for this aspect of the instruction.

Developing functional thinking

One of the most significant changes for the students was adapting to how the technology permitted the user to 'define' functions by rule and domain specifications, and then perform operations on these. The following examples illustrate the syntax of some commonly used procedures.

- Define $f(x) = 1/x$
- $d(f(x), x)$
- $\text{ComDenom}(1/(x+2) - f(x)/(4-5x))$
- $\text{Solve}(f(x)=0,x)$

Students quickly learnt that these operations could be combined for computational efficiency, such as

- $\text{Solve}(d(f(x),x)=0,x)$
- $\text{Zeros}(x^3-2x^2 - 13x + 12) | x > 0$
- Define $f(t) = e^{at}(t-b) | t \geq 0$
- $\text{Limit}(f(x),x,3)$
- $\text{Solve}(f(2)=3 \text{ and } f(-1)=-5, \{a,b\})$
- $\text{Solve}(\int(a/x, x, 0, 2)=1,a)$

The use of the 'composition' feature, where students can pass the results of one function to another, was a major change in the way that students have normally worked. In the past, students would have used more steps, that is, successive operations to achieve a desired solution. Now students are combining operations, minimising re-entry and reducing the number of steps required. As an example, observe the range of methods to solve the problem as shown in Table 1.

Table 1

<p>Problem: Find the value of the parameters a and b if $(x + 1)$ and $(x - 2)$ are factors of $ax^3 - 44x^2 + bx - 12$</p>	
<p>Method 1 Let $p(x) = ax^3 - 44x^2 + bx - 12$ Find $p(-1) = 0$ and $p(2) = 0$ by hand Solve the two resulting equations by using solve(equation 1 and equation 2, {a,b})</p>	<p>Method 2 Define $p(x) = a*x^3 - 44x^2 + b*x - 12$ Evaluate $p(x) = 0 x = -1$ Evaluate $p(x) = 0 x = 2$ Solve the two resulting equations by using solve(equation 1 and equation 2, {a,b})</p>
<p>Method 3 Define $p(x) = a*x^3 - 44x^2 + b*x - 12$ Evaluate $p(-1) = 0$ Evaluate $p(2) = 0$ Solve the two resulting equations by using solve(equation 1 and equation 2, {a,b})</p>	<p>Method 4 Define $p(x) = a*x^3 - 44x^2 + b*x - 12$ solve($p(-1) = 0$ and $p(2) = 0$, {a,b})</p>

What was interesting about this example was the variety of methods students used — all ultimately successful, but with differences in their efficiency and degree of composition. It also struck me as interesting that some students could quickly understand Method 4, while other students needed to be led towards this method more slowly. Also, it was pleasing to note that students suggested that (as in many previous cases) it was important that you first work with by-hand methods, otherwise it may be difficult to understand. As the year progressed, when these students were attempting similar problems, they moved towards favouring Method 4 in this type of problem.

We were often surprised how quickly students learnt the syntax, and adapted to most of the idiosyncrasies of the calculator operations. Often students delighted in revealing to the class another discovery that they had made about a short cut or new method, or explain why a particular procedure might not work. For example, a student was able to explain why $f'(1)$ could be calculated using $d(f(x), x) | x=1$ while $d(f(1), x)$ would not work.

Students differed greatly in how they documented their solutions, and many students work reflected the challenges in demonstrating in writing their understanding. We all struggled with this: students used a combination of CAS steps and by-hand/head steps in obtaining their solutions, and some used calculator syntax in their written solutions. Members of the project team assisted by encouraging use of the Reason-Input-Procedure-Answer (RIPA) method for students when documenting their solutions (Ball & Stacey, in press).

Equivalence of form

A major theme emerging from our experience in the project was how students coped with the variety of algebraic forms presented in the calculator's symbolic output. It was not trivial for our students to interpret this output. Many of the results calculated were not in a form that was expected, in some of the following aspects:

- the order of terms;
- the degree of factorisation;
- the placement of negative signs;
- the reduction of surds;
- the display of fractional components;
- the degree to which terms are combined.

Part of the challenge here was for students to discern the calculator's conventions for algebraic simplification. This was particularly problematic in tests, where multiple-choice questions often required extra work to make them 'look like' the available options in the 'stems'. Students needed to develop methods for rearranging the output, or recognising that it was an equivalent form to the expressions they had expected.

In this respect, students with a stronger by-hand/head algebra seemed to adapt to these problems more effectively.

Of particular interest was how solutions to trigonometric equations were treated. For example, if the student entered $\text{solve}(\sin 3x=1/2,x)$, the CAS answer was $x=12(@n1+5)\pi/18$ or $x=12(@n1+1)\pi/18$.

The presence of an integer parameter (e.g. @n1) was somewhat confusing, and even less helpful as you could not assign this parameter any values on the CAS.

While the general solution was interesting, most students were not impressed with the 'solution generator' syntax of the result, and in most instances chose to restrict the domain so that all solutions were explicitly displayed.

Changes during the project

As noted earlier, students adapted quite quickly to the calculators, and were pleased that most of the functionality of their previous calculator was integrated into the CAS calculator. All students improved greatly over the two years in using the CAS effectively, and became more strategic in their use of it. Table 2 summarises the general trends in their usage over the period.

Table 2

Issue	Early student responses characterised by...	Later student responses characterised by...
<i>Removing existing definitions for variables and functions</i>	Frustration Belief that calculator was malfunctioning	Clearing existing definitions before each new problem
<i>Order of operations</i>	Wrong answers Unchecked Inadequate brackets	Checking of input Clear, even overuse of brackets
<i>Understanding error messages</i>	Frustration Repetition of Entry Lack of understanding of cause	Greater understanding of cause
<i>Use of symbol manipulator</i>	Single steps on CAS Rewriting of entries/results	Multiple steps on CAS Use of cut/paste Definition of functions Use of defined function notation in subsequent steps Encapsulation/Composition of steps
<i>Remembering syntax</i>	Frequent error messages and repeated attempts to correct syntax Inefficiencies in inputs	Reference to procedural syntax notes Most syntax committed to memory
<i>Equivalence of form</i>	Frustration/Bewilderment Little understanding of the CAS simplification conventions	Improved ability to 'see' varieties of form Improved ability to use CAS to change form Greater understanding of the CAS simplification conventions
<i>Recording work</i>	Use of calculator syntax in written solutions Lack of cohesion in documentation of solution	Less use of calculator syntax in written solutions More cohesive documentation of solution

Reviewing course content

Throughout the project, as teachers we encountered a number of topics in which we wondered about the value of the existing emphasis on procedural techniques. For example, how much practice of long division of polynomial was necessary or useful, and for what reason? We were able to obtain good advice from members of the project team (including the VCAA) about these matters, but it seems clear that some topics and emphases will need to be reviewed in light of this project if CAS calculators are to become more widely integrated into school calculus courses.

What we noticed was that our students were on the whole more confident than past groups about the procedural calculus routines, which is not surprising as the emphasis on mastery of manipulative procedures had eased.

However, while that may have eased, the introduction of more abstraction, a broader range of functions, and more combinations of functions provided the new 'hard work' for students (and teachers). We also grappled with 'new' problems where the language was somewhat unfamiliar. Some of these questions were provided by the research team, and of an experimental nature, such as potential new examination questions. In some of these, students were asked to make a conjecture based upon patterns in CAS output. We will get better at teaching with these as we see more of them, but as teachers we appreciated the deeper conceptual basis of many of the newer questions.

The need for students to be able to formulate algebraic problems did not appear to change – they still need to be able to turn a set of statements into variables and equations. Our students were not better or worse at comprehending worded questions than previous groups, or in deciding what the appropriate variables were and which procedures may be the most appropriate. There is however some hope for this area if we can spend more time 'mathematising' than we have previously.

As a tool for learning, the CAS calculator did permit me to spend more time asking students to reflect and choose what procedures to apply, and less time actually applying the procedure. In most algebraic procedures, the higher level task of 'choosing' is repeatedly interrupted by the lower level task of 'applying'. Perhaps, as Kutzler contends, there is hope that using a CAS allows the student to more fully concentrate on the higher level tasks of choosing and responding to the feedback on the correctness of each choice (Kutzler, 1999).

Conclusion

The availability of CAS calculators for students presents many challenges for secondary mathematics education, arguably more than graphics calculators did. Some of the lessons may be the same as for graphics calculators. For example, it appears that a student's facility with each new manipulative routine strongly impacts the degree to which they are able to effectively use (and make sense of) the CAS symbolic routines. It may also help them establish when it is quicker to complete the procedure by hand.

At a school system level, this project heralds an exciting opportunity to review what manipulative skills and concepts we truly value in our senior algebra and calculus courses, and opens up the possibility of focussing on newer, deeper problems. It may be that some routines are so vital to conceptual understanding that we should continue to expect students to perform them in high stakes assessment tasks.

Finally, we should not lose perspective about the influence of the CAS. The presence of the calculator is one of many variables that impacted on student understanding and performance. For instance, the style of expected external exam questions was a significant factor in determining the way that students and teachers in the project used the CAS. Further, the importance of student's comprehension of the question wording, and efficiency in completing the solutions remains critical to their success.

As in other studies involving CAS, other variables such as student attitude, teacher beliefs and behaviours are also critical, perhaps more so than the CAS (see for example Kendal & Stacey, 2001; Tynan & Asp, 1998). While it is tempting to think that the impact of CAS calculators will be revolutionary, our experience in this project suggests that changes are more likely to be slow and evolutionary.

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Lessons from variation research I: Student understanding*

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Although the chance and data curriculum would not exist without variation — variation in random processes and variation in data collection — the topic of variation itself has not had a high profile in the chance and data curriculum. If variation is the foundation of chance and data, what do school students understand about the concept and how do they imagine variation to occur in random processes? This session will present a summary of research findings for students in grades 3–9, as well as the outcomes from interviews with some 6-year-olds that indicate children can discuss variation from an early age.

It is now a decade since chance and data emerged as a formal component of the mathematics curriculum in Australia (Australian Education Council (AEC), 1991, 1994). Initial research into students' understanding of topics in the curriculum focussed on concepts such as probability, graphs and their interpretation, average, sampling, and comparing two data sets. Reports have been made to teachers on some of the outcomes (e.g. Watson, 1999). Arising from this research and the suggestions of others (e.g. Shaughnessy, 1997) has been the realisation of the necessity for understanding of the variation that underlies the topics in the curriculum. Except for a brief mention of standard deviation in Band D, variation gets no specific mention in the *National Statement* (AEC, 1991). Without variation, however, the chance and data curriculum would not exist and research is now focussing on the variation that is built into the other more explicitly mentioned topics in the curriculum: variation as part of chance processes; variation as observed in graphical representations and its meaning; and variation in sampling. Explaining variation is not always simple but the intuition that variation exists is very strong.

The only way to study variation is through the other topics and hence current research is an extension of that done previously, building on tasks that were originally rather closed in nature, to include speculation about and predictions of variation. In some interview situations, it is also possible to experiment with random processes and ask

* This paper has been subject to peer review.

students to react to the outcomes. Gaining insight into students' reactions can help teachers plan classroom intervention activities.

The descriptions of student understanding presented here are based on the observed outcomes from over 800 surveys completed by Tasmanian public school students in grades 3, 5, 7, and 9 and interviews with 73 students: 7 in preparatory grade, 18 in each of grades 3 and 5, and 15 in each of grades 7 and 9. About 300 of these students took part in a unit of study on chance and data emphasising variation. The outcomes of the teaching experiment were positive and are reported elsewhere (Watson & Kelly, 2002a, 2002b, 2002c). This report will focus on the development of understanding displayed with respect to early ideas on variation expressed by 6-year-olds, variation in relation to chance, variation shown in graphs, variation in sampling, and the expressions for meaning of the term variation.

Early ideas on variation

The interviews of the preparatory students included questions related to drawing 10 lollies from a container of 100, 50 of which were red, 20 green, and 30 yellow, and to a story about students recording the maximum temperature in Hobart every day for a year and finding the average to be 17°C (Watson & Kelly, in press). In both contexts the children had strong intuitions about variation, change, or differences in potential outcomes. In predicting the number of reds, in a handful of 10 lollies, some used a phrase like 'about 5', and all students in predicting repeated trials of 10 displayed variation in the numbers chosen. Although not appreciating the importance of the proportion of red lollies in the container, several reasoned with ideas like 'if you shake them you might not get the same amount.'

For the questions based on the weather scenario, all students appreciated variation in the temperature over time, with comments like 'the temperature changes' and 'one day it might be cold; the next day it might be colder.' Although again sometimes having difficulty with giving reasonable values within maximum and minimum ranges, they all predicted temperatures for six different days of the year. One demonstrated variation by drawing activities appropriate for summer and winter. Overall it appears to be realistic to discuss variation and change in many contexts in early childhood, with conversations leading to establishing criteria for reasonable predictions. How many red lollies out of 10 would surprise you? Why? Would 30°C be a surprising temperature in January? In July? Why?

Variation and chance

After tossing fair dice a few times, children appreciate the fact that different numbers come up. The random process involved leads to the expectation of 'different outcomes' but some uniformity if many trials are performed. Hence the word chance is used to describe the equi-probability of the outcomes of tossing a die. 'It's just chance' is also a phrase used, however, to describe outcomes that stray from the equi-probable expectation. It is precisely this tension between the expectation based on theory and the variation from it that exists in practice that is at the heart of the chance and data curriculum. This is succinctly expressed by David Moore when he says, 'Phenomena

having uncertain individual outcomes but a regular pattern of outcomes in many repetitions are called *random*' (1990, p. 98). The question is, how much variation should be expected in a random process before one suspects something is wrong? Previous research (e.g. Green, 1986) has shown students to be relatively poor at suggesting appropriate variation in random processes and not improving as they get older. Hence it is important to start discussion not only with questions like, 'Which is more likely: a 1 or a 6, when a die is tossed?' and 'Why?', but also with tasks to suggest a reasonable distribution of outcomes from 60 die tosses. For many grade 3 children, creating a set of outcomes adding to 60 is too difficult but by grade 5 the adding task is manageable for most. Of interest then are the distributions produced and the reasoning that accompanies them. Often reasons are quite idiosyncratic but open up opportunities for discussion. Although not totally clear the following grade 3 response attempts to explain a very lop-sided suggested distribution.

'15, 11, 10, 5, 4, 2 — Because on the 6 there are 6 holes, so there is more weight on the 6 compared to the 1. If the 6 had bumps the 6 would be rolled more often.'

Other responses show a range of reasoning and expectation.

'9, 12, 11, 10, 10, 8 — They are my favourite numbers.'

'10, 10, 10, 10, 10, 10 — Because they all have an equal chance.'

'10, 5, 7, 8, 10, 20 — Because you would not know which numbers you will get.'

'9, 11, 10, 10, 8, 12 — Because each number should come out roughly 10 times each in 60 throws.'

The technical distinction between *the* 'most likely outcome' in a theoretical sense and *a* reasonable outcome displaying variation, is one that probably should be left to the senior secondary level, but classroom trials by students should be convincing on two scores: after many trials, outcomes approach the expected proportions and yet there is 'always' some variation from these expectations.

Many other opportunities exist using coins, spinners, and drawing objects from containers, to reinforce ideas of expected pattern on one hand and deviation from it on the other.

Variation and graphs

There are many issues associated with both displaying and interpreting variation in graphs; again, if there were no variation in data sets it would not be necessary to draw graphs. Moritz (2002), for example, shows graphs created by primary students and the variation they put into their representations to tell a story. The issue discussed here is related to students' perceptions of the information a graph contains and how it can be used to make inferences or predictions.

Among the most frustrating questions for students are questions like parts (d), (e), and (f) for the 'travel graph' shown in Figure 1. Different from many mathematics questions, these ask for a prediction that goes beyond reading the information in the graph. Acknowledging the uncertainty of choosing among several options and yet venturing a suggestion is considered the most statistically appropriate response.

Although almost all children think that the graph will show variation from day to day (part (c)), many cannot translate this uncertainty to predicting outcomes. The range of responses to part (d) reflect story-telling, inability to decide, the boy-girl dichotomy, balancing the boys, noting a pattern, observing the frequency of girls who come by car, and including uncertainty.

‘Boy: we know because of his voice.’

‘I don’t know. It can be either.’

‘I don’t know. Because it is 50-50 chance.’

‘Boy: because there is only one boy who goes by car.’

‘Girl, she’s the last person on the graph.’

‘Boy, because there is a pattern, 2 girls, boy, 2 girls, and then a new boy.’

‘Boy, because then there would be an even class, 14 girls and 14 boys.’

‘Girl: because more girls go by car than boys.’

‘Girl: it could be a girl because most of the people that come by car are girls.’

The explanations for part (e) often reflect the context that children imagine for the class.

‘There might not be a track leading to the school.’

‘Nobody lives very far away.’

‘No one goes to school by train.’

The most statistically appropriate response, however, reflects that change could occur.

‘Nobody went to school by train that day.’

Again for part (f) many students assume that Tom was ill the day before, influencing their choice of transport. Those using the information in the graph make two reasonable suggestions based on the graph but again acknowledging uncertainty is the most appropriate response.

‘Car: because he normally walks but he hurt his leg.’

‘Walk, he lives close to the school.’

‘Bus: because more people go by bus.’

‘Bike, because that is the most common for a boy.’

‘Probably by bus, because one third of the children caught it today.’

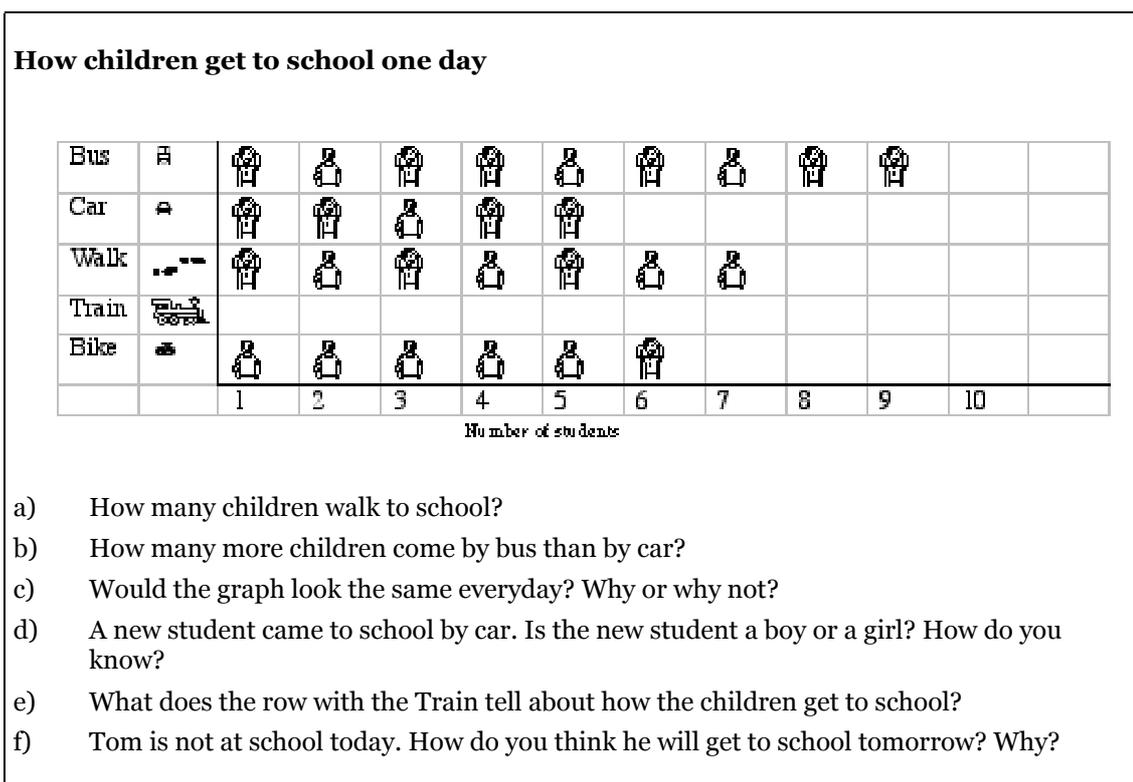


Figure 1. Travel graph task.

Another aspect of graph interpretation is the appreciation of the scale that is used if frequency of numerical values is being reported. Figure 2 shows two stacked dot (or line) plots that were created by two groups of children in an actual class to display how many years the families in their class had lived in town (Watson & Kelly, 2002d).

In telling the story represented in each graph there are four levels of increasingly sophisticated response.

‘3 is the biggest family.’ [Misinterpretation]

‘Someone stayed there for 10 years. Someone stayed there for 1 year.’ [Data reading]

‘That most people have stayed in town for different amounts of time. Four people have stayed for 3 years.’ [Data reading plus summary]

‘More families lived in their town around the 4/5 year mark. But 10 and 15 years is quite popular.’ [Two summary statements]

Appropriate note of variation both in terms of reading the data and interpreting the information in context is an important feature of graph-reading that people encounter all through life. Less than 20 percent of students overall choose Graph 2 as a better way to tell the story of how long families had lived in the town, accompanied by appropriate reasoning.

Graph 2, because it shows each year individually.

Graph 2, because [it] is more visual and you can notice the time difference in how long each family has been there more precisely.

A class of students recorded the number of years their families had lived in their town. Here are two graphs that students drew to tell the story.

Graph 1

a) What can you tell by looking at Graph 1? [2 spaces provided]

Graph 2

b) What can you tell by looking at Graph 2? [2 spaces provided]
 c) Which of these graphs tells the story better? Why?

Figure 2 Stacked dot plot task

Variation and sampling

Sampling from a population is an example of the ‘part/whole’ concept that occurs across the mathematics curriculum. As a sample gets larger it gets better at representing a population but how large a sample should be to do a good job is not an easy question. As well a sample needs to be chosen in such a way that it is indeed representative and not biased in one way or another. The idea of fairness in a statistical sense may conflict with children’s ideas of fairness. Hence the surveys of students asked how they would carry out a survey of a school of 600 children, 100 in each of grades 1 to 6. It also asked for reactions to several proposals by ‘other students,’ which included drawing 60 names from a hat, asking all grade 1 students, surveying 10 members of the computer club, asking 60 friends, and setting up a table outside the lunch room. The responses to the original questions cover an expected range.

‘All of them.’

‘450.’

‘I would survey 28. I would choose my friends and some other people.’

‘300 — 50 from each grade.’

‘60 — picking out of a hat or a race.’

Increasingly with grade, students detect bias in the other methods suggested. Reactions to choosing 10 members of the computer club include the following.

‘Bad: it doesn’t give many people chances.’

‘Bad: because he is only asking 10.’

‘Bad: because if they were in a club they would probably be best friends or like the same things and say the same as each other.’

The conflict with children’s ideas of fairness arise mainly in two scenarios. Some disagree with picking 60 from a hat because some grades might not get chosen. Others agree that having a table outside the lunch room was good because then students could choose whether they wished to participate or not.

Further conflict with respect to sampling was observed during the lessons carried out with primary children as part of the research. They were happy with the random selection of a sample of five children to represent the class. After the process was repeated several times, however, and a particular child was chosen randomly a second time before all other children had been selected (‘had a go’), this was not considered fair. Although most students have a basic understanding of the term sample and its purpose, much work with sampling and chance processes that create variation will be required to reinforce appropriate ideas during the middle school years.

Variation — the word

Variation itself is not a word that is easy to define, particularly out of context. To assist students the survey asks a three-part question:

- (a) What does the word variation mean?
- (b) Write a sentence using the word variation.
- (c) Name something that varies.

Responses to these questions are classified by the reasonableness of the three answers. The third question is the easiest. The second often causes trouble because ‘variation’ is a more difficult word to construct a sentence around than ‘variety’ or ‘vary’. Several sets of answers demonstrate the range of responses.

- (a) ‘It means a variety of something.’
- (b) ‘There is a variation in the weather today.’
- (c) ‘Weather.’
- (a) ‘Variation means all different stuff and not the one thing.’
- (b) ‘Variation in picking the stuff.’
- (c) ‘Rolling a dice, the answers vary.’
- (a) ‘There are different types of the same thing.’
- (b) ‘The variation of soccer balls is huge.’

- (c) 'The numbers in a Tattslotto draw varies.'
- (a) 'Change something a little.'
- (b) 'Variations were made to the actors script.'
- (c) 'Weight.'

Although these questions are not easy, the responses point out the need for discussion of terms like variation, random, and sample. As well as discussion, students need to gain experience writing about the ideas, particularly in the context of classroom activities of data collection and inference (see Watson, 2003).

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Supporting the development of mathematical thinking*

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Preliminary research indicated that UWS pre-service primary teachers lacked confidence in teaching mathematics and, in particular, problem solving. Also, UWS students in both primary and secondary areas experienced difficulty in obtaining expert and quick mathematical pedagogical advice while completing their in-school professional placements. Mathematics specialist lecturers were unable to supervise all their secondary students, while primary students required advice from many specialists. This has occurred in a context of shrinking university budgets, where in some cases specialist lecturers were replaced by much cheaper non-specialist part-time staff. This paper presents activities and strategies that were used to support and enhance the development of mathematical thinking of pre-service primary and secondary teachers of mathematics. The paper highlights and discusses the importance of two web sites designed specifically for this purpose and the mathematics teaching and learning issues associated with these sites.

At the University of Western Sydney, I present mathematics education courses to both primary and secondary pre-service teachers. The aim of these courses is to develop the mathematical thinking of pre-service teachers while addressing the pedagogical issues associated with the development of mathematical thinking of their potential school students. A key assumption of the courses is that everyone can do mathematics once they have been engaged. The course also reflects the international trends in mathematics education such as:

According to today's standards for mathematics education, problem solving, communication, and mathematical thinking are the overarching skills required by modern life. Technology can provide the tools for applying these skills (Jarrett, 1998).

Many educators believe that ICT (information and communication technology) offers the most benefits to students and teachers when it is integrated into the core curriculum, rather than considered as a separate discipline. Thus it is important for pre-service teachers to be immersed in the efficient and effective use of ICT during

* This paper has been subject to peer review.

their pre-service courses in order to give them the confidence and the knowledge to incorporate ICT in their own teaching. Changes are continually occurring to the way teachers operate and the way teachers see the world. So the UWS courses seek to help the pre-service teachers make best use of the rapid development of ICTs, even while change is happening around them. They are given the opportunity to consider carefully the social and individual effects due to ICTs. So while discussing the strategies used in the development of mathematical thinking this paper will also address the questions: What does technology contribute? How can technology support learning? Can you teach the same content without technology? Is technology more efficient and effective?

All courses are presented using a combination of lectures and tutorials. Students have access to a website for each course, which is found on the *Blackboard* platform of the University's server. This website supports the development of mathematical thinking using a variety of means. The first involves the availability of lecture notes, tutorial activities and copies of the overheads used in the lectures to students to download and read before the lectures. This removes the drudgery of students copying and provides time for the students to engage in and do some problem solving while exploring their thinking. It allows the lecturer to conduct an interactive question and answer style of presentation and helps the students to deepen their understanding by making connections with prior learning. Students who are unable to attend are able to minimise their loss by having access to the notes.

Tutorials are very practical and involve the students in activities that develop thinking and develop strategies for teaching mathematics with a problem solving perspective. All students are required to have a mathematics problem solving exercise book. Every tutorial begins with a problem which each student attempts individually for ten minutes, then shares with another, then with the table group. Finally the strategies for solving the problem and the pedagogical implications are discussed in a whole class setting. The problem, solutions and lesson strategies are recorded in the front of the workbook and become a great resource for the pre-service teachers during the school service. The back of the tutorial workbook is taken up with a number of extended investigations, such as the *Mystery Bone* (Clarke, 1996).

The tutorial involves other activities linked to the teaching of topics chosen from the syllabus. These activities are chosen from a range of 'best practice' publications (e.g. Gould, 1993; Lovitt & Clarke, 1988, 1989; Lovitt & Lowe, 1993). Students are also expected to purchase a tutorial resource book which has material that has been found to be useful over a long period of time and in a variety of situations and is explored in tutorials. While in many cases the activities can be considered 'best practice', they do not constitute a 'magic' or 'fool-proof' prescription for teaching. The material contained in the book could be considered as contributing to the knowledge required of professional teachers of mathematics. As professionals, teachers are always searching for and collecting good ideas and activities that promote the thinking and learning of their students. Some of the activities are games and the reason for including these is because:

Games should not be regarded merely as a useful activity when teaching and learning have been accomplished, but should be seen as an integral part of a balanced program. They are most effective when structured around mathematical

ideas and when playing is dependent on mathematical understanding. In this way, they provide a context that is real to children as they become fully engaged in something in which the outcome matters, leading to the realisation of the value of the underlying mathematical processes (Booker, Bond, Briggs & Davey, 1997, p. 19).

The website is supportive of claims that the Internet can promote more collaborative forms of teaching and learning (Jefferies & Hussain, 1998). The discussion site on the course website does this while enhancing mathematical thinking. At the very beginning of the course, students are given one week to sign in and 'say hello'. This is a check that all students have access and the necessary computer knowledge and skills. In the first week students are divided into learning circles. Each learning circle is given a private chat room on the discussion site. The use of learning circles linked via the Web reduces the need for meetings and allows the students to gradually build and refine their ideas as the course progresses. At the end of each week of classes, one or two questions are displayed on the *Blackboard* site. The learning circles are asked to reflect upon and discuss how to answer the questions. This helps the students focus their thinking on the key ideas for that week. One component of the course assessment is a two hour end of semester examination that consists of five questions designed to test knowledge and understanding of the content and methodology covered during the course. These questions are taken from those displayed each week. One of the gains from this assessment strategy is that the students' stress from fear of the unknown in examinations is removed, or at least greatly reduced. Another gain is revealed in the quality of the answers, which exhibit a deeper understanding of the issues as a result of the prolonged thinking and reflection. Learning circles using the Web disagrees with the claims of others that the Internet is an isolating technology that reduces participation in communities (Nie & Erbring, 2000).

Another excellent use of the discussion site is a space for the lecturer to answer questions. It is a rule that questions concerning assignments, examinations or course material will only be answered via this space. This ensures that every student has access to the information. It also saves the lecturer a great deal of time.

The development of mathematical thinking continues even while the pre-service teachers are on their professional placements. This was a result of research conducted with the pre-service teachers. A secondary group who had already completed one placement period were surveyed. The two questions reported here from the secondary pre-service teachers group (21 students) survey are: 'What is your greatest frustration when on practicum?' and 'How could I help you during practicum?'. The results were surprising to me.

There were two strong common responses to the first question. The secondary pre-service teachers reported their greatest frustration was having a tertiary supervisor who did not have a mathematics teaching background. At UWS, because of various constraints, a methods lecturer is assigned to a couple of schools and supervises all the students in that school. Thus the lecturer has students from all the key learning method subjects such as English, Science, and so on. Perhaps the lecturer supervises two or at most three mathematics methods students during a professional experience period usually of four weeks duration. Lately, because of financial pressure upon universities

in Australia, casuals are replacing lecturers as a cheaper option. These casuals are usually retired teachers from across all key learning areas. Thus a student is fortunate to have a supervisor with a mathematics background.

The second popular frustration of the secondary pre-service teachers was the difficulty in getting assistance when a situation became difficult. In follow-up conversations, while students were appreciative of the help given by the other teachers in the school, this help was not always available due to the busy nature of the job. The secondary pre-service teachers were also reluctant to raise certain issues through fear of 'losing face' in front of the other teachers. They were trying to impress the classroom teachers in the hope of being offered casual or full-time employment when they completed their course. In the eyes of the secondary pre-service teachers, there was a lot at stake by appearing to be a proficient and competent teacher. As a logical consequence, results from the second question indicated a strong plea for access to expert support.

I then surveyed the primary pre-service teachers who had completed two school placements. Three of the questions used with the primary methods students (97 students) were: 'What is the most difficult key learning area you have to teach while on practicum?', 'What is the key learning area you are least confident to teach while on practicum?' and, 'How could I help you during practicum?'.

For the first question, the majority of students regarded Science then followed by Mathematics as being the most difficult to teach during the professional experience sessions in schools. The order of these two subjects was reversed in the responses to confidence in teaching, with the primary pre-service teachers indicating they were least confident in teaching mathematics followed closely by Science. The answer to the third question mirrored the responses of the secondary group in that students wanted access to advice when needed that was independent of the school.

In response to this data, and inspired by a research paper (Herrington, Herrington & Omari, 2000) I successfully applied for a teaching grant during 2000 and began constructing the site using the *Blackboard* platform on the University's server. The site (called MAPS for Mathematics Assistance to Practicum Students) contains a number of features which contribute to the development of mathematical thinking. The first feature provides access to 'best practice' material such as collections of classroom teaching ideas and lesson plans across all year levels, and topics of the mathematics syllabus. Some of the grant money was spent hiring 'recognised expert teachers' to write material for this feature of the website. The New South Wales State Education system has a number of syllabus documents that cover Kindergarten to Year 12. Currently the syllabus documents for years K–10 are under review but at the time of writing this paper the following are the mandatory documents for NSW teachers. The primary syllabus covers K–6 (NSW DOSE, 1989) and consists of three main strands: Number, Measurement and Space. While it contains many useful teaching strategies, there are areas that lack depth of treatment. The early secondary syllabus covering years 7 and 8 (NSW BOS, 1988) contains the strands of Problem Solving, Geometry, Measurement, Statistics, Number and Algebra. There are three syllabus documents in years 9 and 10 for the three courses of Standard, Intermediate and Advanced Mathematics (NSW BOS, 1996a, 1996b, 1996c). In years 11 and 12 there are four courses: General Mathematics (NSW BOS, 1999); Mathematics (formerly called 2

Unit); Mathematics Extension One (formerly called 3 Unit) (NSW BOS, 1997a); and Mathematics Extension Two (formerly called 4 Unit) (NSW BOS, 1997b). Obviously providing material for all these courses is a mammoth task and it is still in progress, yet what is available is very popular according to the number of recorded hits each week on the website and the unsolicited comments and emails from students. Classroom teachers have also requested access to the site for the purpose of obtaining ideas and resources for their planning and teaching.

Another feature of the site was the provision of an annotated guide of links to other helpful websites. Students are encouraged to submit reviews of websites, which are then checked by other students. If there is agreement then the new site and the corresponding description is added to the list. This is an area that is too large for the lecturer to monitor and allowing students to control this feature of the website has proved very successful to date.

The second most used feature after the resource collections is the threaded discussion facility. It is here that students can post a request, provide comments, and responses from the lecturer are also posted. Other students may contribute advice or comments. This facility is monitored and controlled by the lecturer who can remove unwanted or insensitive remarks before they are displayed. The site has a protocol for online discussion that was adapted from others that are available. It contains the usual advice such as:

Start with a friendly greeting and use a friendly and relatively informal tone. You would never go up to a stranger and blurt out a question!

Ask yourself 'Would I say this to their face?' before you send the message. Re-read your message before sending/submitting. It can be embarrassing and often impossible to recover careless messages or those with glaring spelling mistakes. Humour, if brief, can be a great asset in getting messages read.

It also contains the usual 'don'ts' such as:

Don't use all upper case letters (IT MAKES IT LOOK AS THOUGH YOU ARE SHOUTING!!!!).

Don't use discriminatory language (e.g. sexist or racist).

During the practicum periods, the lecturer monitors this facility regularly and responds either via the threaded discussion facility, a personal email or a by phone if necessary. For example one student from the secondary program wrote:

I had my first lessons and they were so terrible! It was 2 x 50 min lessons for yr9 adv math class. Kids were OK when they were with my supervising teacher, but in my lesson they just didn't listen and I had serious problems in attracting students concentration on the tasks and controlling the whole class. So my supervising teacher gave me 1/14 - total failure!

After discussion with my supervising teacher, I found I had problem with

- * Long teacher talk.
- * Disorganised lesson (My lesson went astray very quickly!)
- * Need to be more assertive.

* Lack of teacher-students interaction and the list goes on...

I agree with everything my supervisor commented, but I felt so down after my first shot. I feel so bad. Can you give me some tips that can improve my lessons? Also I would like to know how long teacher's talk should be. (i.e. what's the kid's concentration span?).

This student had recently migrated to Australia from an Asian country and was on a steep learning curve concerning NSW schools. UWS is one of the institutions servicing the NSW Department of Education and Training campaign to attract new teachers by funding retraining and accelerated university courses in Mathematics and Science teaching. In this case a telephone call was made to the student and time was spent allowing the student to ventilate his feelings before gently moving him to consider his part in the situation. I will not include the advice and strategies suggested but they were in harmony with the maxim that 'if you manage learning well, you don't have to worry about managing learners' (Guskey, 2000, p. 59). The issue arising from this example is in the importance of speed in providing feedback. The student was able to return the next day with confidence and with a plan of what to do. He managed to pass his professional experience after a very poor start.

To complete the tour of this website, there are two other features such as the frequently asked questions, and an entry port for submitting material. The frequently asked questions mainly dealt with the administration of professional experience. However, some of the threaded discussions were summarised and then added to the list of questions and answers. The entry port allowed students to submit material that was vetted by the lecturer. This also contributes to the gradual accumulation of resources and ideas.

Conclusion

At the beginning of this paper, the following issues were raised in relation to the development of mathematical thinking of pre-service teachers: What does technology contribute? How can technology support learning? Can you teach the same content without technology? Is technology more efficient and effective?

This paper discussed the construction of websites as a means of developing mathematical thinking and supporting the needs and concerns of pre-service teachers. Technology has greatly contributed to the conduct of the UWS courses as well as facilitating access to expert and quick mathematical pedagogical advice to pre-service teachers during their in-school experience. This use of ICT resonates strongly with Burghes's (2002) observation:

The latter part of the 20th century has seen phenomenal advances in available technology. The catalyst for these advancements has come from industry and commerce, but education has been able to exploit the potential of this ever advancing technology to enhance teaching and learning (p. 49).

This paper has also argued against comments made by critics such as Koblitz (1996) who wondered whether more could be achieved by putting resources into simple materials that would stimulate students to think more and to develop their imagination and creativity. In order to present these courses without technology in the current UWS

environment, a greater use of time and money would be required. In the near future, it is unlikely that there will be an increase in either, so the challenge then becomes finding further means by which ICT can be used to enhance the thinking of the pre-service teachers. The MAPS website is both dynamic and adaptable and there is the possibility of universities cooperating and combining expertise for the benefit of all their students. A really creative solution by the universities in my state of New South Wales who offer teacher education in mathematics would be to pool and share their mathematics education expertise. It is hoped that this paper also challenges teacher educators of other key learning areas to consider implementing similar initiatives in their subject area.

To summarise, ICT has allowed a more efficient and effective provision of courses and professional experience assistance to pre-service teachers. The success has been overwhelmingly positive if measured against the feedback received from the students. While there is still considerable development work to be completed, students regard it as a valuable resource:

I have just spent numerous hours downloading resources from the Practicum site on Blackboard to use next semester when I go out casual teaching for the Catholic system. I am, as you can probably understand, very nervous about teaching Mathematics and these resources will come in very handy in easing these nerves of mine.

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Explaining success in school mathematics: Mythology, equity, and implications for practice⁺

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The theme of this conference allows considerable scope for papers and presentations. I have read the theme ‘Making Waves’ as giving license to opening up the discourses and practices of school mathematics to radical critique; in other words, to make waves within our discipline, to challenge many of the taken-for-granted practices, and to uncover some of the alienating practices that are entrenched within the discipline. By undertaking this task in the keynote, my intention is to raise awareness of the problematic nature of contemporary mathematics education and to suggest some possible solutions. Hopefully some of the issues raised in this paper will resonate with experiences of participants. Combining both research and my own experiences across a large range of classrooms, I see the task of this keynote as to raise educators’ awareness of how practice is implicated in the construction of students’ identity and their participation in school mathematics. More important is the consideration of equity: when considering the outcomes of teaching in mathematics, there is a need to consider equity as well: who is represented in the top (and bottom) groups of outcomes? Are the outcomes equally distributed across brackets of gender, social background, cultural background, geographic location, or language background? How do students feel about their learning experiences? These are just a few of the questions that need to be posed. If there is not an equal distribution, why is this the case? Mathematics is strongly implicated in the legitimisation of social divisions.

The current testing regimes that are part of contemporary schooling — e.g. the Year 3, 5 and 7 benchmarks — provide some valuable insights into trends across states and the nation. Similarly, international studies such as TIMSS and PISA allow comparisons of where Australian mathematics education sits in relation to other countries. The results of these tests can be challenged on many grounds, but it is not the intention of this paper to do so; rather, it is to consider the general trends within these outcomes. It is clear that there are problem areas in the teaching of mathematics. In reporting on the PISA study, the authors (Lokan, Greenwood & Cresswell, 2001) note that Australia

⁺ Invited keynote presentation.

performs quite high on numeracy levels in comparison with other OECD countries, but the gap between high achieving and low achieving students is much greater than in comparable countries. Students from low socio-economic backgrounds performed significantly worse than their middle-class peers. Numerous reports have indicated that indigenous students perform significantly lower than national averages. The focus of this paper is to explore reasons as to why this is the case, and how is mathematics education implicated in the construction of these differences.

A simple explanation: Ability

One of the most pervasive myths in education to explain difference in outcomes is the notion of ability. Most explanations of different outcomes are premised on the innate notion of ability: students perform well because of ability, or conversely, perform poorly due to lack of ability. Of all the discipline areas covered in schools, mathematics is heavily imbued with this discourse. For example, in a large study of UK teachers it was found that when asked about ability and ability grouping, most teachers did not subscribe to the view that there was a need to group students according to ability in subjects other than mathematics. In this curriculum area, more than 80% of the respondents reported that there was a need to ability group in mathematics. This begs askance of why, if the notion of ability is innate, is it that only in mathematics is there such a preponderance to subscribe to this discourse. Why is there not the subscription to ability group in English or Science or SOSE? Are the students not differentiated in these areas as well? Why is ability such an important discourse in mathematics but not the other areas of schooling? What are the effects of ability grouping? This is the first issue with which I argue: my position is that the notion of ability in mathematics is a mythology, albeit a mythology to which teachers and the wider community over prescribe.

Teachers' beliefs: Pygmalion in the classroom

In a seminal study (Rosenthal & Jacobsen, 1969), students were assigned to a new teacher. The students were introduced to the teacher with randomly assigned grades against them; the teacher then interacted with the students for the duration of the year. At the end of the year, the final results of the students correlated with the initial random assignment of grades. This study shows the power of teachers' beliefs about students. Where the teacher assumes something about a student's capacity, she/he will interact with that student in particular ways, thus producing particular outcomes. The interactions can be social – in terms of how the teacher talks with the student; praises/rewards the student; expectations of the student – through to the cognitive demands placed on the student. Such a study could not be repeated in today's research climate, but it is interesting to note should similar situations occur in today's classrooms, particularly in mathematics.

In a large study of UK classrooms to identify the characteristics of an effective numeracy teacher, the authors (Askew, Brown, Rhodes, Johnson & Wiliam, 1997) identified ten key identifiers. Of these, two key points are relevant to this paper. The first is that teachers must hold a fundamental belief that all students can be successful

in their study of mathematics. This finding relates closely to the previously cited study showing that teachers' beliefs about learners is a strong factor in determining whether or not students will be successful. Believing students could learn and be successful influenced teachers' actions and outcomes.

The power of the individual teacher cannot be downplayed. In a large study of Victorian literacy classrooms, (Hill, 1994; Hill & Rowe, 1998), the authors claim that it is not the school that makes the difference in student performance, but rather the individual teacher. As such, teachers have an important role to play in the success of their students.

In light of these studies, it becomes important for student learning that individual teachers hold beliefs that students can learn mathematics — by holding and then adopting practices that will foster. Perhaps one of the biggest challenges to student learning in mathematics is the notion of ability, and the subsequent subscription to the practice of ability grouping.

Ability grouping in mathematics: Equity implications

Perhaps one of the most pervasive practices in mathematics is grouping students according to some perceived ability. In some countries, such as the UK, the practice is entrenched policy and schools group students around ten levels. Tests are implemented and students are assigned to various levels. Text books are devised around the various ability levels (Dowling, 1991; 1998). In Australia, the practice is less rigid. However, underpinning the practice of streaming, setting or ability grouping is the belief that students have an inherent mathematical ability and that by assigning students to levels that correspond with this perceived ability, the mathematics can be better tailored to the level of the student.

This seemingly apolitical practice raises a number of questions: why is it that the different ability levels seem to be populated by particular demographics in the students? Apple (1996) has argued that the lower streams are occupied by students from working-class and disadvantaged backgrounds. This is reinforced by studies in the UK (Boaler, 1997; Cooper & Dunne, 1999). However, social class differentiations are difficult to ascertain from Australian data, although Teese (2000) has shown such differentiation in the study of senior mathematics. When considering one of Australia's most disadvantaged groups, it is most often Indigenous students who are in the lower ability groups in many remote and rural schools. This begs askance as to why the lower streams are disproportionately over-represented with students from disadvantaged backgrounds. Again, the simple solution is based in eugenic arguments whereby such groups are seen to have lower intellectual functioning, and hence their location in such groups. Such an overt argument is untenable in today's society but the implicit compliance with the practice of ability grouping and the correlation with social, cultural and linguistic background suggests its unproblematic acceptance.

In the remainder of the paper, I would like to take issue with ability as a construct, propose some ideas as to why social differentiation occurs, and to offer alternatives. In the first section, I challenge the idea that ability grouping reflects some innate capacity. Rather, I will propose that the practice can *create* differences between students. The

different outcomes of ability grouping may not be due to some innate ability, but the practice of ability grouping itself; that is, ability grouping reifies differences. In the second major section, I draw on various practices in mathematics and demonstrate how they subtly recognise cultural, social and linguistic differences so as to support particular attributes of students and marginalise others.

Ability grouping in mathematics: The chicken or the egg

The first main issue that I call into question is that of ability grouping. It is a highly pervasive practice in mathematics. Interestingly, there is little data to support the general belief that ability grouping supports all students by providing curricula relevant to their levels. Where there is limited support for ability grouping, it is for the students in the highest stream. However, these reports are challenged: Boaler (1999), for example, found that even in the upper streams, there is not full support for learning at that level. Overall, ability grouping disadvantages most students, but mostly those in the lower groups. In contrast, research in heterogeneous groupings show greater support for learning for all students at all levels. In spite of this sustained body of research across most Western countries, the practice remains endemic in mathematics classrooms. If prolonged research has called into question the value of ability grouping, the question has to be asked as to why it persists in mathematics.

Drawing on data where students in Years 9 and 10 were asked about their experiences of school mathematics, it became clear that the comments were bipolar. Those in upper streams reported mostly positive experiences, a strong self-concept of their mathematical ability and the usefulness of mathematics. The converse was true for students in lower ability groupings. In terms of making waves and challenging the status quo, I will only draw on a minimal number of quotes to bring forward the issues. This is to illustrate how the practices of school mathematics support and marginalise students, and how the practices are implicated in the production of outcomes of low achievement rather than to demonstrate empirically the points being made; that task is for another forum.

Teacher beliefs about ability of students

In terms of the students' perceptions of their teacher's beliefs in them as learners of mathematics, the comments are poignant. In most cases, where comments of this kind were offered, they were of the ilk that the teachers had little confidence in their students' potential to achieve, and in many cases the students cited that they felt that their teachers held quite negative beliefs about them. This is evident in the comments below:

Thomas: In this class, all the dumb kids just are here to muck around. The teacher thinks we are dumb and doesn't really care too much about what we do. (Beechwood, Year 9)

Tyler: I don't like being in this class 'cause it is the only one I feel dumb in. I mean in English or workshop, I am doing OK, but in maths, I feel like a retard. The teacher treats us as if we know nothing. (St Michaels, Year 9)

Megan: It is like they say, 'You are smart and you are dumb,' and then put us in classes where they [the teachers] make sure it happens. (Huon Pine, Year 9).

The comments here raise the issue of teachers' beliefs about students' ability. These comments represent those offered by students whereby they recognised that their teachers thought they were 'dumb', 'stupid', 'idiots' and the subsequent positioning of students. This is particularly evident in the comment offered by Megan where she recognises the effects of this labelling through the practices of teaching. Converse comments were offered by students in the upper streams where they saw the teachers seeing them as 'clever', 'smart' or 'intelligent'.

Assessment

Students commented that the assessment practices reified their positioning within mathematics. All students sat common year level tests throughout the year. The content to which students were exposed was differentiated, thus effecting potential outcomes in tests. Students (in both streams) recognised that the upper streams had covered all content contained in the exam and went into the exam already with a pass grade: the purpose for them in the exam was to enhance their grades. In contrast, the lower streams had only covered some of the content and many of the concepts in the exam were unfamiliar, thus excluding them from working with that content. Their unfamiliarity of the content meant that these students went into the exam fighting to gain a pass grade and with no hope for gaining anything higher. Students from the lower streams offered comments such as these:

Simon: In our revision for the test we only cover the easy questions as the really hard questions are for the smart kids. They don't even have to do the process [easiest] questions on the exam as they already have got an SA [pass] for maths so they can go straight to the hard ones. We don't even get taught that stuff so can't even do it on the exam. The most I can get is a SA. (Pine Bark, Year 10)

Mel: I like doing the easy work, but it is not fair when it comes to the exam 'cause we don't even know half the stuff that is on it. We haven't even covered it so we are lucky just to pass. (Huon Pine, Year 10)

However, the comments were not restricted to the lower streams. The students in the upper streams offered comments that supported these claims.

Randolf: What is good about this class is that I already know I have passed the exam. I just have to work on the harder questions to get an HA or VHA [higher marks]. The kids in the dumb class don't even cover the hard stuff so they can only pass the exam. (Melalucca, Year 10)

When seen in this light, it becomes little more than a self-fulfilling prophecy. Students are placed in classes where they are exposed to different curricula, offered the same assessment, and the outcomes of 'objective' testing practices 'prove' the 'ability' of the students. As such, questions need to be posed as to whether the outcomes of the tests are true indicators of students' ability, or whether they cause the outcomes they produce? What is being tested: students' mathematical ability, or the curriculum covered.

Locked out of upgrading

Some students articulated their frustration with this system. They were keen to demonstrate their capacity to move out of the lower classes, but were thwarted on two fronts: the assessment as noted above, but also the daily practices within the classroom were problematic. As noted by nearly all students (and teachers) in this study, the ethos of the lower streamed classes tended to be more chaotic and off-task than the upper streams. The issue of behaviour management in the lower streams was a strong feature noted in the study. Students wanting to move out of these classes were stifled by the lack of engagement by their peers, thus restricting the content that could be covered. The comments offered below reflects this aspect of the inability for upward mobility, and thereby the locking out of progress.

Rachel: I get so annoyed with maths 'cause I want to get out of this class. The teacher doesn't really care about us, the boys all muck around and we get no work done. I have worked really hard and even got a home tutor to help me. But we aren't doing the hard stuff so I don't know the work on the exam. I want to get into a better class but I just can't. (Beechwood, Yr 9)

Rachel's comment was echoed by a number of students — most often girls — who were very frustrated by the practices that were effectively locking them out of progress.

Classroom ethos

As noted in the above section, teaching in these classes represented a challenge for many teachers and when working through the transcripts, the candid comments offered by the students indicated how the students could manipulate the teachers and classes. Boys, in particular, were quite open in their criticisms of the teaching and assessment practices, and about how they would openly thwart the teacher's actions and actively choose not to engage in the lessons.

Mohammed: Well, me and my mates just sit there with our books open and don't do any work. Like, we talk about stuff we did last night or who we were talking with on the chat lines or games we played but we don't do maths. I don't want to be there and the teacher doesn't neither so why should we have to go. (Beechwood, Year 9)

Brad: We hate being in maths. The stuff we learn is just crap. The teacher doesn't give a shit about us so we don't want to be there. We set him up all the time. Like he wants to be our buddy so we ask him about cars and stuff so he raves on about that stuff and doesn't do any maths. (Beechwood, Year 10)

Emmet: I'm in the dummies class. It is really boring in there so I see it as my job to make a bit of fun. Me and my mates will set the teacher up. He has got a short fuse so it doesn't take long. I get sent off to the Thinking Room but at least that is better than maths. (Huon Pine, Year 10)

These comments suggest that the students in the lower streams contributed knowingly and willingly to the ethos of their mathematics classes. In some comments, this was due to their resistance to the issues discussed earlier: assessment, teaching practices, teachers' beliefs, and their need to be in stimulating environments.

While these comments are only brief in terms of the overall project, they give some sense to the lived realities of the students. The objective and subjective structuring practices position students in ways that reify the perception of ability. The formal measuring practices associated with assessment legitimate their location in the ability-groupings. The more subjective practices further enhance perceptions of their perceived ability. Through these processes — teaching and assessment — the students have internalised their 'ability' as evident in their comments about themselves as learners of mathematics. This is evident in the language they have used to talk about themselves and their classes. Throughout the transcripts, terms such as 'clever', 'smart', and 'bright' were used in relation to the upper streams whereas terms such as 'dumb', 'stupid', 'idiots', and 'retards' [sic] were used in relation to the students in the lower streams. This research indicates that the practice of streaming premised on the notion that classes are structured so as to support learning, may not be doing this. Rather, they may be contributing to the lower grades of students, thereby reifying notions of ability. As such, the practice of ability grouping needs to be called into question since it may be contributing to many of the problems being faced by contemporary educators.

Rigidity in school practices

In this section, an alternative program to mainstream schooling has adopted practices that are different from traditional programs. Students reactions to these programs are noted. In one of my community roles, I work with the local TAFE. As part of this work, I am involved with students-at-risk programs. These are typically students who have left school with little education; who have been forced to leave school and are undertaking study; or students who are on government support and need to work on their skill base. Working with this cohort of students is admittedly a biased sample but their insights into school mathematics are valuable learning episodes as they provide insights into what is problematic. For many of the students in these programs, they experience success for the first time, in part due to the specific tailoring of programs to meet their needs. The higher degree of flexibility in teaching is also valuable. Being less constrained by school regimes, the students are able to walk out of the room to calm down when they become frustrated — as opposed to being sent to a behaviour management room and the potential for exclusion. Students noted the rigidity in school structures as highly problematic. When facing new mathematics content, they would become scared and threatened and react in adverse ways. This was seen as problematic in the school setting and often punished. Such punishment, under the guise of a behaviour management program often enticed students to rebel or thwart the system of management. In contrast, the ethos in the TAFE sector recognised these issues for young people so that when they were frustrated, they were able to walk outside the classroom, sit down on a bench, take some time out, have a cigarette if necessary, and come back to class when ready. When they were back in class, they were on task, and their behaviour did not restrict the learning of their peers.

- Robert: Here it is better. I can do my work at my pace. If I don't understand anything, there are plenty of people who want to help me. If I get angry, I am allowed to go for a walk and calm down. I can then come back into class and just go on with my work.
- Marcus: I hated maths at school. We had to sit down and do sums for an hour or more. That is so boring. Here we do an hour or more of maths, but I can get up and walk around. I work at my pace and when I need help, I can ask for it. I can pick and chose what I want to do. If I feel like a lazy day, I pick easy stuff. If I feel like a challenge, I take some harder stuff. I don't have teachers bossing me around. This is how I learn best and I like it here.
- Justin: The good thing about being in TAFE is that you can chill out when you need to and the teachers aren't on your case the whole time. I think that they better understand us and where we are coming from. In my school they just were looking for excuses to get me out so when I did anything, it would be down to the Principal. It was not fair but I knew what they were doing and I couldn't win. I hated maths at school and my teacher hated me so it was just a battle. Here they are prepared to give me a chance. I want to learn and am doing really well. It is a nice feeling.

The comments offered here are indicative of the greater flexibility that can be afforded in alternative settings. Students' comments reflected this need for greater freedom and autonomy when in learning situations. The rigidity of school structures may not suit all students and for many of the 'at-risk' students, this is possibly even more the case. The comments offered by TAFE students suggest that where there is this greater freedom, they are better able and willing to learn. The oppositional culture of youth is well documented in the literature on youth studies, so the constraints imposed by contemporary schooling may be a source for resistance and subsequent marginalisation. The comment offered by Marcus above, indicates the less restrictive format of TAFE maths lessons is more in line with his style of learning.

The comments offered by the students to date, indicate a number of areas where the practices of school mathematics may be contributing to the lack of success of many students in mathematics. While the data here have not been able to offer insights in the construction of social differences in school mathematics, there are ways in which this is done in subtle ways. These will be discussed in the following sections.

Mathematics as 'The Sabre Tooth Curriculum'

Students across a range of studies have raised problems with current teaching in mathematics. Perhaps one of the most common issues raised by students was that of relevance. As one of the guiding rationales for the inclusion of mathematics and its important status within the curriculum is that it is a highly relevant and important life skill. If this is the case, then it should also be the case that links should be possible with other discipline areas and the world beyond schools. Much like the seminal paper *The Sabre Tooth Curriculum* where the importance of knowledge to the wider society is seen as an essential element, students questioned the relevance of what they learnt in

mathematics to their worlds beyond (and within) schools. In the case of some students (generally those in the upper streams in secondary school), there were some links being made:

Alfredo: We get the best teachers [in the upper stream]. They are really good. Our teacher was an engineer so he can make us understand why we need to do maths. (St Michael's, Year 10)

In contrast, many of students could not see the relevance of their mathematics. Indeed as indicated in the comment below, when students asked as to 'why' they needed mathematics, there was little to appease them:

Jodie: I often sit in class and wonder why we do this stuff. In my other subjects, I can see the point but not in maths. One day I asked the teacher why we had to learn this stuff and she just yelled at me and said 'because'. (Melaluca, Year 9)

Rachel: I find maths so useless. I mean, when will I ever use or need this stuff we are learning. I can't see me ever using it. My teacher tells us we have to learn it 'cause it's good for your brain. I don't see that as much of a reason! (Huon Pine, Year 10)

Most students saw little purpose in what mathematics they learned and why they were doing it. As such, links between mathematics and other areas of school and beyond are needed. However, it is important that consideration is made of the ways in which relevance is developed.

Contextualising mathematics

One of the big moves in mathematics education — to make the links between school mathematics and the world beyond school — has been the inclusion of mathematics in context. However, with the use of contexts, there is a need to recognise the types of contexts used: at a very superficial level is the pseudo-context, where there is a veneer placed over the mathematics in order to justify its role. In contrast, there are problems where the context is as authentic as possible. It is the latter version that has greater credibility when considering relevance.

While links with and across disciplines and the world beyond schools are important, some caution must be taken in terms of how these links are made. In his detailed study of UK textbooks, Dowling (1991; 1998) showed that lower streams were given examples that were low level and focussed on activities undertaken by the working-class. In contrast, the examples in the higher streams involved complex reasoning and focussed on tasks undertaken by the upper and middle-classes. This class analysis of work and mathematics has been important in indicating how the streaming of mathematics classes correlates with social background. Dowling's thesis showed the process through which class labour was directed by indoctrination via school mathematics texts, as well as providing an ideological critique of the assumptions that underpin mathematics teaching. This work is highly relevant in considering examples that are used within mathematics classes and how these can be used to subtly position students in relation to career aspirations.

In a large study where we (Zevenbergen, Sullivan & Mousley, 2002) are exploring the use of open-ended questions in mathematics, one of the key issues we have found as a barrier to learning is that of context. The contexts that are used for embedding mathematical tasks must be culturally relevant and sensitive. In seeking to develop open-ended tasks, it has become a large feature of this study that many of the tasks need to have contexts that make sense and have relevance to students (Sullivan, Zevenbergen, & Mousley, 2002) if students from disadvantaged backgrounds are to be successful in undertaking mathematics.

Language and contexts

It has been shown (Zevenbergen & Lerman, 2001) that when using contexts, consideration of the language imposed by the contextualisation process must be recognised. Using Bernstein's notions of linguistic codes, students from disadvantaged backgrounds may not be able to 'crack the code' (Zevenbergen, 2000) of the mathematical register and will reply incorrectly because they have misidentified the register in which to respond. For example, tasks asking students to calculate how many four-litre cans of paint are needed to paint a room where dimensions are provided require the students to identify the mathematical discourse, and then calculate the appropriate answer – which in this case involves rounding the answer up to the nearest whole number. This type of questioning may appear to make the question relevant to the real world. However, it now involves further dimensions closely related to equity (Zevenbergen & Lerman, 2001). For some students, accessing the appropriate discourse is not always obvious. In their large study, Cooper and Dunne (1999) reported that middle-class students are more likely to identify the problem as mathematical and hence respond appropriately. In contrast, students from working-class backgrounds (and other backgrounds where they do not speak middle-class English) are more at risk of identifying the problem as an everyday problem and respond in this discourse. Thus the contextualisation of mathematical problems may create as many problems as it attempts to address. Being aware of this potential area of access is important: a correct response by students may be due to their capacity to code-switch between school and everyday discourses, rather than some innate ability. The differences observed in student outcomes in tests, such as those studied by Cooper and Dunne, may not be due simply to ability but are a function of some other variable – one which is not explicitly taught in mathematics. Creating contexts for mathematics problems needs to be considered more carefully. In creating a context, does the context represent a real problem or is it creating a veneer that legitimates the mathematics as if it were a real problem?

Authentic pedagogy in mathematics

In attempting to redress the issues that have been raised to date in this paper, there are a number of possibilities. While the data presented here is not extensive, it has been used to create a situation that highlights the need for reforming mathematics education for contemporary schools and the world beyond schools. Issues to do with teachers' beliefs about learners and the subsequent learning environments created for those

students, relevance, contexts, and teaching approaches have been shown to be problematic for some students, particularly students from disadvantaged backgrounds.

Contemporary youth studies show that students in today's society are quite different from their peers. Terms such as 'Gen X', 'Gen Y' or 'Cyberkids' have been employed to discuss the uniqueness of contemporary youth. Many of these differences relate to learning in a multi-input society. Unlike previous generations who were 'trained' to think in a linear fashion due to the presentation of written text, young people process information differently due to multiple stimuli and processing. It is not uncommon to see young people playing video games, watching television, listening to the stereo on high volume and carrying on a conversation on a mobile phone. This multiplicity of information input and processing significantly differs from other generations. Multi-tasking in a technology-rich environment has been a way of life for many young people. Similarly, there is a greater sense for immediate feedback. This suggests that many of the teaching approaches that were part of an earlier era may not have the same application in today's classrooms. In concert with understanding the changing needs of young students, there is the parallel need to consider contemporary work patterns. As I have argued elsewhere (Zevenbergen, 2001), patterns of work are changing dramatically, as are the skills needed for contemporary workplaces. Questions need to be posed as to the value of mathematics taught in schools and its links to the world beyond schools. This is particularly the case when considering the highly technologised society within which we live and how such technology impacts on ways of working and thinking mathematically. The potential impact of the calculator on mathematical thinking has been well documented (Ruthven & Chaplin, 1997). Its resonances with the workplace cannot be underestimated.

In terms of the issues raised by students in the earlier sections of this paper, it becomes important to consider how best to organise practices that will meet the needs of students, prepare them for their lives beyond schools, and to ensure that they are mathematically literate when they enter their post-school years. How this can be achieved is through many of the dimensions articulated through Education Queensland's *New Basics*' productive pedagogies. Considering the four dimensions of the productive pedagogies, many of the issues associated with developing learning environments that will cater for the diversity in mathematics classrooms can be addressed. These four dimensions are listed in Table 1 below.

Table 1

Intellectual quality	Relevance	Recognition of Difference	Supportive School Environment
deep knowledge deep understanding substantive conversation knowledge as problematic meta language	knowledge integration background knowledge connectedness problem-based curriculum	cultural knowledges inclusivity narrative group identity citizenship	student control social support engagement explicit criteria self regulation

In considering how these elements of productive pedagogies can be developed, a case study will be used to illustrate.

A case study

A small NSW town, suffering from the economic hardships facing many rural towns, was offered the prospect of employment through a large national company. The company proposed dumping Sydney waste in one of the valleys surrounding the town. The Mayor put the proposition to the Council with the recognition that the offer would create much needed jobs for the town. A group of primary school students from the local school were concerned about the environmental impact that the dump would have on their surroundings. They undertook an environmental impact study and questioned the actual employment opportunities for the town. They developed a report, presented it to Council, and defeated the proposal.

One only has to look at Table 1 to sense that many of the elements of productive pedagogies are evident in such an activity. In order to undertake such a project, the students needed considerable mathematical knowledge — that is, deep knowledge and understandings of mathematics. The teachers involved were able to develop a meaningful and purposeful context for the work being undertaken — with real outcomes for the students. This context resonated strongly with the students' lived experiences. The mathematics that was needed had relevance and application to the tasks being worked through. The levels of engagement were significant as the impact of the initiative had real consequences for the students. This type of activity allowed teachers to engage the students in a substantive learning experience that had considerable impact on their lives. Furthermore, it made mathematics a very important and relevant area of study.

Concluding comments

Through this paper, I have attempted to alert readers to some of the problems confronting contemporary mathematics educators: some are more problematic than others, some are taken for granted but are still highly problematic. The students' voices used in this paper indicate their lived experiences. When so many students are opting out of mathematics, some of the insights provided by the students in this paper, provide educators with insights into the experiences on the 'other side of the desk'. By considering pedagogy in a critical manner and recognising the problems faced by students when learning, and their subsequent sense of identity in relation to mathematics, it becomes evident that some practices need to be called into question if students are to engage in the discipline. The use of productive pedagogies' frameworks gives teachers a language of description by which they can develop critique and reform in teaching. Many teachers are already engaging in this process and there are considerable examples of excellent practice. This is undeniable. However, more needs to be done — particularly for those students most disadvantaged by schools and mathematics. The issues raised in this paper seek to bring some of these to the fore if mathematics educators are to redress the inequities so strongly evident in practice and outcomes. As I began this paper, and I reiterate, teachers' beliefs about teaching,

learning and students have the most significant impact on the outcomes for their students. By believing that all students can learn and then developing practices that are appropriate for students living in 'new times', the chances for success are greatly enhanced.

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Workshops

Teaching the big idea of 'statistical significance'

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Introduction

The big idea of 'statistical significance' is a key concept that is taught to enable students to grasp some of the essential ideas in inferential statistics. Statistical significance is a fundamental concept behind the objective of performing inferential statistics. As variation occurs with data we need to judge at what point do we move beyond the natural expected variation with our sample from a population to then be able to make the judgement that a 'real' change has occurred in that population. Then we are able to draw conclusions about that wider population from which we sampled. Generally sample data is generated from either a questionnaire or a series of measurements. In most instances we want to see if a particular manipulation has caused a significant change or to establish some characteristic about the population from which the sample was chosen. It is important that the teacher explains the related concept of variation and the learner, in order for the concept of 'significance' to be taught effectively, appreciates it. At the Auckland University of Technology, statistical significance is presented over a wide range of differing contexts within a series of degree programmes. Our experience has shown that it is hard to teach and generally students have difficulty grasping this key concept. This makes teaching this concept problematic and thus gives rise to a series of differing teaching techniques being used. In many cases different techniques have been used with the same class over different semesters. My workshop will use the *Minitab* package to examine in more detail some of the eight different techniques that I have used in teaching this key concept.

How do we teach this big idea of statistical significance?

The eight different teaching techniques, which I have used at AUT, are described below, along with what is problematic about each technique. Most of these techniques are used with our non-mathematically inclined students in business, food science, nursing, journalism and sports and recreation degrees. These techniques are wide ranging in terms of their mathematical content. The first five involve calculation and deductive reasoning whereas the last three involve deductive reasoning only.

1. Traditional

This technique is very much based on a mathematical process. As part of this process definitions are required for hypotheses, level of significance, errors, p-value, population parameter and critical values. Formulae are required for test statistics. Decision rules are formulated along with a conclusion. A rigid four to six step process is applied depending on the course. It is possible for students to just follow rules by computing the test statistic compare it to a critical value and learn which hypothesis to conclude without really understanding about significance. For example if the test statistic is greater than 1.96 conclude the alternative hypothesis or if they are looking at some output from a statistics package provided the p-value is less than 0.05 conclude significance. Students just learn the basic format of the null and alternative hypotheses along with if the p-value < 0.05 we have significance. In many cases students miss the idea of variation and how a p-value, which as it becomes smaller represents further departure from the expected variation of the population parameter under consideration. It becomes difficult for students to transfer their knowledge to other situations and in most cases they end up learning a set of rules in order to cope. Any understanding of significance gets lost in remembering the definitions along with the steps of the various processes. In many cases students look up a formula from a supplied list and substitute into that for starters.

2. Tail probability

This technique follows on from teaching students to look up probabilities. It was an extension of our food science statistics course where students had to use tables to find probabilities. For example, a typical exam question would be like:

Filled containers with a label weight of 200 g are checked at the production line.
What sample weight would be needed to conclude that the label weight had significantly increased?

A sample is selected and a value of 220 g is obtained. To see if 220 is far enough away from 200 to conclude that the mean of 200 has increased significantly, we calculate the probability that the weight is at least 220 g given the standard deviation. This probability can be regarded as a p-value and a significant increase is concluded if $p < 0.05$. The 'big idea' is that we have moved into the tail of the probability distribution and moved outside the expected variation range of the weights. In using this technique several curves can be sketched so students can observe tail probabilities at the upper end. Another key example is looking at taste panels with such questions as: 'How many panellists out of 25 have to detect the modified product before we conclude that the modification is significantly detected?' In many cases there exists tables with cut-off points. In the main this technique is rule based, students have difficulty in transferring this concept to a variety of problems and by using specially constructed tables the visual display of 'significance' is missed. Students need to get a grasp of the probable variation in a variety of situations and how we move outside that when $p < 0.05$.

3. Confidence interval approach

This technique came out of an extension of descriptive statistics. Based on a sample a range of probable values for a population parameter like the mean is constructed in the form of a confidence interval. To give the students a taste of inferential statistics we then claim that μ has some value. If it lies within the computed confidence interval the claim is verified otherwise it is refuted. In Black (1996), we used this approach with several of our nursing classes where we needed to examine whether there was a significant difference between two independent groups. A confidence interval for the difference in the two group means was computed and we were interested whether that interval contained 0 or not. The key concept for the student to grasp was that the confidence interval represented the very likely range of values for the difference in the two means. Once we stepped outside that range we had 'significance'. These techniques are rule based and it is possible for students to apply them without really understanding 'variation' and what a confidence interval represents. Also available teaching time for the nurses became a factor. Black (1998) testifies that students' knowledge transfer from this technique to understanding 'significance' with p-values in a research article was found to be difficult to achieve. We can extend this to two independent means with either overlapping confidence intervals or not. With our journalism students on comparing two proportion estimates we are looking for a gap of at least two margins of error before we conclude significance. There is still some probability of this happening involved, as there is some chance of being outside the margin of error for each proportion estimated. The concept of students stepping outside the expected variation is well illustrated with this method. Again students can apply this method without really understanding 'significance'.

4. Control Charts

A good starting point is some real data that contains variation. We construct a chart with a specified centre line along with warning and action lines. Ishikawa (1976) explains that it is expected that 95% of the values will lie within $\mu \pm 2\sigma$ provided the process is operating as expected. If a value lies outside 3 sigma (action) limits (probability = .003) or two successive points outside 2 sigma (warning) limits (probability = $0.05 \times 0.05 = .003$) we have significance and some action is usually taken. For example a fill volume above the upper action line would imply reducing the fill of a bottle. As a prelude students have had instruction in constructing such charts, plotting points, which are usually 'means of samples' along with background work on the normal distribution. The link with 'significance' comes when we step outside the expected variation of the plotted mean or range values. This technique is linked to a real problem throughout with the related concept of variation being demonstrated in many practical settings. However these practical settings are limited to statistical process control situations. Students are able to transfer their knowledge into proportions by using fraction defective control charts. However it is difficult to transfer into situations involving differences between two or more means. In addition the control chart technique is limited to courses involving some types of quality control.

5. Operating characteristic curves

Using this technique involves the visual representation of the probability that a lot has acceptable quality based on a sample plotted against probable lot quality. Ishikawa (1976) describes this visual presentation as an *operating characteristic* (OC) curve where the null hypothesis is taken to be that ‘the lot has acceptable quality’ against the alternative hypothesis that ‘the lot has unacceptable quality’. An OC curve can be used to visually represent p-values when we have both good and bad lots. A difficulty with this technique is that variation within a sample is unclear from the curve and students have difficulty in transferring the concept of ‘significance’ from this very specific application into other situations.

6. Correlation

A series of scatter plots can be linked to a series of correlation values. Students come to appreciate what ‘correlation’ measures. Students perform the exercise of matching correlation (r) values to a particular scatter plot as described in Baumgartner and Strong (1998). Testing the null hypothesis of ‘no correlation $r = 0$ ’ against an alternative of significant positive correlation $r > 0$ or significant negative correlation $r < 0$ students can be taught to appreciate what happens to the scatter of points as correlation becomes significant. So somewhere between 0 and 1 the value of r becomes significantly positive and most computer packages would associate a corresponding p-value with an r -value. This works the opposite way to how significance is shown with the other techniques, particularly with control charts, in the sense that the more the points line up, the less variation from a fitted line and hence the more significant correlation. As r increases the p-value decreases until $r = 1$ and $p = 0$. When fitting a normal distribution, we have the opposite sense when the points line up, we conclude a good fit and the p-value is > 0.05 as the null hypothesis is being concluded.

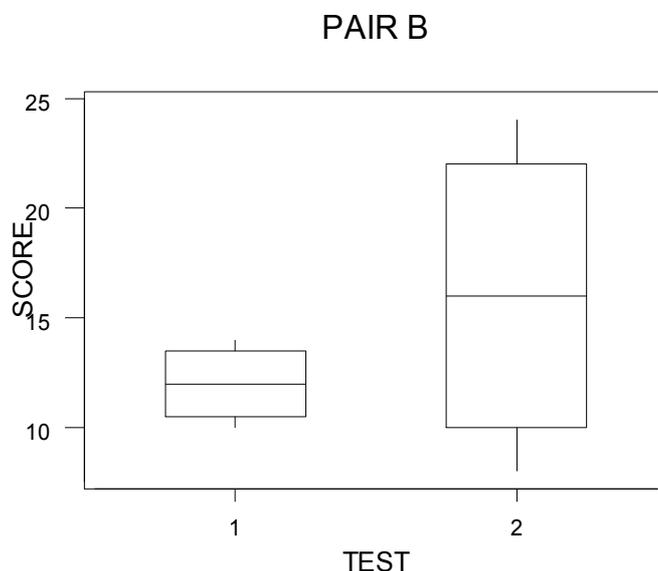
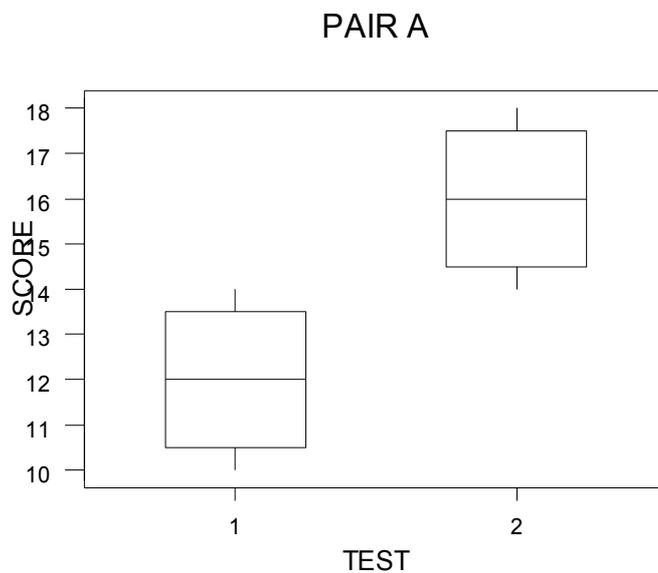
7. Various means case study

I would set this case study scene with a practical example. For example suppose a nurse averages 8 hours sleep per night. If a sample of nurses yielded a mean of H hours sleep per night would you conclude that u , the mean was now significantly less than 8 when H equalled 7.5 hours, 7 hours, 6.5 hours or 6 hours. Students give a ‘gut feel’ type answer. Ideas such as variation can be discussed along with the p-value being thought of as a measure of the ‘distance’ from 8. Students can be brought to the realisation that there has to be a cut-off point between us concluding $u = 8$ or $u < 8$. $H = 6$ hours is clearly $u < 8$ and $H = 7$ hours is clearly $u = 8$. However our nurses are required to interpret a quantitative research article containing p-values so it is important for them to interpret variation, which leads to significance, in the context of the relevant hypothesis. It is hard to bring in the concept of ‘standard error’ as it is outside the scope of their syllabus.

8. One or Two Graphs

(a) Two box-and whisker plots

This technique involves reasoning with little or no mathematical computation. I ask the question in Black (1998): ‘Which of the following pairs A or B would you expect to show a significant difference?’



The answer is A where the overlap is low compared to B even though the difference in the two medians in each pairing is the same at four. These diagrams provide a visual summary of variation. It asks the students to make a judgement on exactly when we have moved outside the expected variation. With a series of diagrams with corresponding p-values being given some link can be made between overlap and p-value. For example as overlap decreases the p-value decreases and the difference

increases. The existence of a boundary is established and the student learns that when the p-value drops below 0.05 we have a significance difference. In many cases the rule is learnt but with the concept of variation being at least partly appreciated. Also the diagram comparisons involve medians whereas with most of the tests involve comparing means.

(b) One or two distribution curves

This is another version of the previous techniques. Two normal curves are pulled apart and a link is made between the amounts of overlap with a probable p-value. It is easy to observe significance when curves are entirely apart however again it is difficult to establish a boundary in the mind of a student. In the case of our midwifery students we needed to establish a link between a t statistic and a p-value. A single curve was sketched attached to a p-value. Over a series of such diagrams, the aim was to reason as follows:

Increase the t statistic, move outside the expected variation, p drops below 0.05 then a significant increase is concluded. In many cases these students had to read several quantitative pieces of research where t values were quoted alongside p-values. A matching exercise between groups of p and t values proved to be a valuable teaching tool. Another version of this technique is to use two bars and slowly pull them apart to illustrate the movement towards a 'significant difference' between two frequencies.

How do students learn this big idea of 'statistical significance'?

Students respond to the teaching of significance in many varied ways. In many cases it is their first module in statistics at AUT. At school they have been taught statistics in a very mathematical way and their reasoning skills have not been developed to any large extent. The use of technology in statistical packages provides ample opportunity to minimise the rote calculations and to pose several what if type situations to assist students to develop their thinking skills. Interpreting output produced from real data by a statistical package assesses this. Hypothesis tests can be performed with decisions being made which lead to appropriate conclusions. Hogg (1991) illustrates that students need processes in a practical context so they can appreciate related concepts such as variation. For example sampling filled containers on a production line where fill volumes are measured plotted and interpreted with practical actions if required. A case study of a practical situation involving other disciplines linked with statistics provides an excellent learning environment. This illustrates an important part of the teaching process as outlined in Pateman (1989), which is to: 'create value by using real life situations. Students place value on what they understand'. In my experience, many of my students approach statistics in a negative frame of mind. It's my view that one of the contributing factors is a result of their past experiences of having mathematics taught in a non-relevant academic fashion. Students like processes that can be remembered like a set of rules without a context. At AUT we always aim to create a practical context by either getting students to collect or use real data, then doing analysis and finally writing up a presentation of their analysis. With our overseas

students, we have language issues, who find it hard conceptually to master definitions and thus appreciate their true meaning as evidenced by our student evaluations. It is important that we attach meaning all the way through teaching a statistical process and students gain the ability to transfer that meaning to other situations both within and outside of their major discipline. In teaching statistics our students are prepared by first being given the skills before encountering the key concepts like variation and significance.

Conclusion — What is really important?

We need to find the best answers to the question, ‘What do we really want the students to know regarding significance?’. Whatever teaching technique we use, there will be always present disadvantages thus making it problematic. In summary, from my teaching experience, I have found it important to ensure the following when teaching ‘statistical significance’:

- build on students’ past knowledge;
- link all appropriate statistics to assist knowledge transfer;
- concentrate on deductive reasoning and interpretive skills;
- move away from rote calculations and a wholly mathematical approach;
- use technology to change emphasis of both teaching and assessment from traditional approaches;
- create practical contexts;
- develop skills in choosing appropriate statistical analysis alternatives;
- emphasise links with the related concept of ‘variation’.

In the teaching, student learning and assessment we need to be continually relating the concept of ‘statistical significance’ to situations that the student can relate to in a practical way. The overall aim is to put the student into a ‘real life’ situation where the concept of significance needs to be either explored or interpreted. If we do this then we give ourselves the best chance of success in both our teaching and the students’ learning about this big idea of ‘statistical significance’.

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A showcase of lessons: Enhancing learning using a graphics calculator

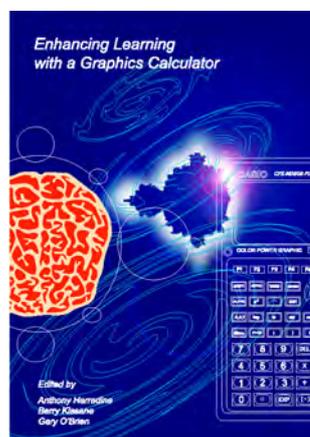
Kiddy Bolger, Merrilyn Goos, Anthony Harradine,
Barry Kissane, & Gary O'Brien

The editors of this new resource and two contributing teachers will illustrate some of the teaching ideas in this book.

The learning of many mathematical concepts can occur in new ways and traditional approaches can be enhanced with the use of graphics calculator technology. This book offers ideas on how a graphics calculator can be used appropriately in the learning and doing of mathematics. The lesson ideas included in this book have been submitted by teachers from across Australia.

The lessons are written to provide teachers with real-life situations through which students are encouraged to explore mathematical concepts. The editors have developed the original lesson ideas further and used them to exemplify the appropriate use of the technology to explore the mathematical concepts and consolidate their understanding. All keystrokes and screen dumps included in this book are based on the Casio CFX9850GB Plus graphics calculator. The publication is also available free of charge on the Australian Casio Education Site (ACES), <http://www.casio.edu.shriro.com.au/>

The resource is also available on the AAMT 2003 conference proceedings CD-ROM.



Creating connections in middle school mathematics

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The activities presented in this workshop grew out of the authors' experiences teaching a mathematics course for middle school preservice teachers in a two-year post-graduate program. They reflect the importance of making connections both among the fundamental mathematical concepts that contribute to the curriculum and between mathematics and other subject areas. Topics covered will be chosen from issues associated with the teaching of algorithms and the use of concrete materials and calculators, problem solving in unusual situations such as testing the claim that a tea bag has '50% more room for leaves to move' or estimating the number of die tosses to fill a 25-hole grid if each toss determines how many holes to fill on the grid, and using newspaper articles to link numeracy across the curriculum.

Introduction

A browse through recent issues of *The Australian Mathematics Teacher* reveals many examples of the ideas of 'connectedness' in mathematics. A discussion of 'sense making' classrooms, for example, suggests that a sense-making culture consists of connected mathematics as opposed to the isolated or disconnected activities in a traditional classroom culture (Flewelling, 2002). Perso (2001) argues for inclusive mathematics teaching that is about finding out where students are at in order to connect new mathematical experiences onto their previous ones. Kissane (2002) laments the fact that although 'official curricula' emphasise mathematics as a way of thinking, the experiences that children have in many classrooms are focussed on a body of knowledge and the thinking processes are often ignored. Each of these ideas about making connections in mathematics is important.

We see mathematics teaching as requiring three kinds of connections, each of which relates to one of the views outlined above. Firstly, there is the need to make connections

within mathematics itself — Flewelling’s ‘sense-making’. If students are allowed to develop mathematical understandings in such a way that these connect to other bits of mathematics, learning is more likely to be deep and retained. Secondly, mathematics teaching needs to connect to students’ prior experiences — an obvious statement but one that can be overlooked in the need to push through the curriculum. Perso’s inclusive approach is relevant here. Finally, there is the thorny issue of ‘relevance’ — connecting mathematics to real world situations. This may become trivial or disconnected, such as using catalogues as a basis for practising an algorithm to calculate percentage. It can, in contrast, become a rich investigative process in which many of the objectives of today’s curriculum documents can be met while encouraging the creative, humanistic, and adventurous thinking that Kissane desires.

How does this relate to schooling in the middle years? No one would argue that approaches emphasising these connections should be restricted to these years. We believe, however, that students in grades 5 to 10 are often turned off mathematics by teaching that emphasises disconnected facts that need to be memorised rather than rich, exciting learning experiences. Mathematical achievement in these years does show a decline, particularly from grades 6 to 7 (e.g. Siemon, Virgona & Corneille, 2001). Many students become increasingly disaffected (Earl, 2000). Making connections in mathematics may well not change the world, but it can improve the experience of mathematical learning for many students in these critical years. In this paper we give some examples of ways that we have used to make connections with preservice teachers who are preparing to teach mathematics in middle schools. All the examples can also be used in middle school classrooms.

Sense-making connections

Is your telephone number divisible by 9?

How can you find out the answer to this problem without actually doing the calculation? Many teachers know the ‘trick’ of adding all digits together and if the total is divisible by 9, then the original number is also. An investigation of this problem, however, provides the opportunity for students to explore and make connections with place value, using calculators and concrete materials.

Before setting the problem, introduce students to Heddens’ (1984) *Numeral Expander*. This allows students to write expanded forms of numbers. The number 7416, for example, is created as $7000 + 400 + 10 + 6$, and, by folding the expander, eight different ways of writing the same number can be obtained. This leads into the division problem posed at the start, using the problem-solving strategy of ‘solve a simpler related problem’.

A demonstration that a number is divisible by nine if and only if the sum of its digits is divisible by nine can be achieved by using Dienes’ blocks and the concept of measurement division. This is not as difficult as one might imagine since measuring off nines can conveniently be done in ‘lots of nine’, such as 99 or 999. The pictures shown in Figure 1 demonstrate how what are left after measuring off many lots of nine from the number 3744 are the units digits 3, 7, 4, and 4. That is, for each of the tens (longs), 1

unit remains after subtracting 9. For each of the hundreds (flats), 1 unit remains after 99 is subtracted. And for each of the thousands (cubes), 1 unit remains after 999 is subtracted. Hence if the sum of its digits is divisible by nine, so is 3744.

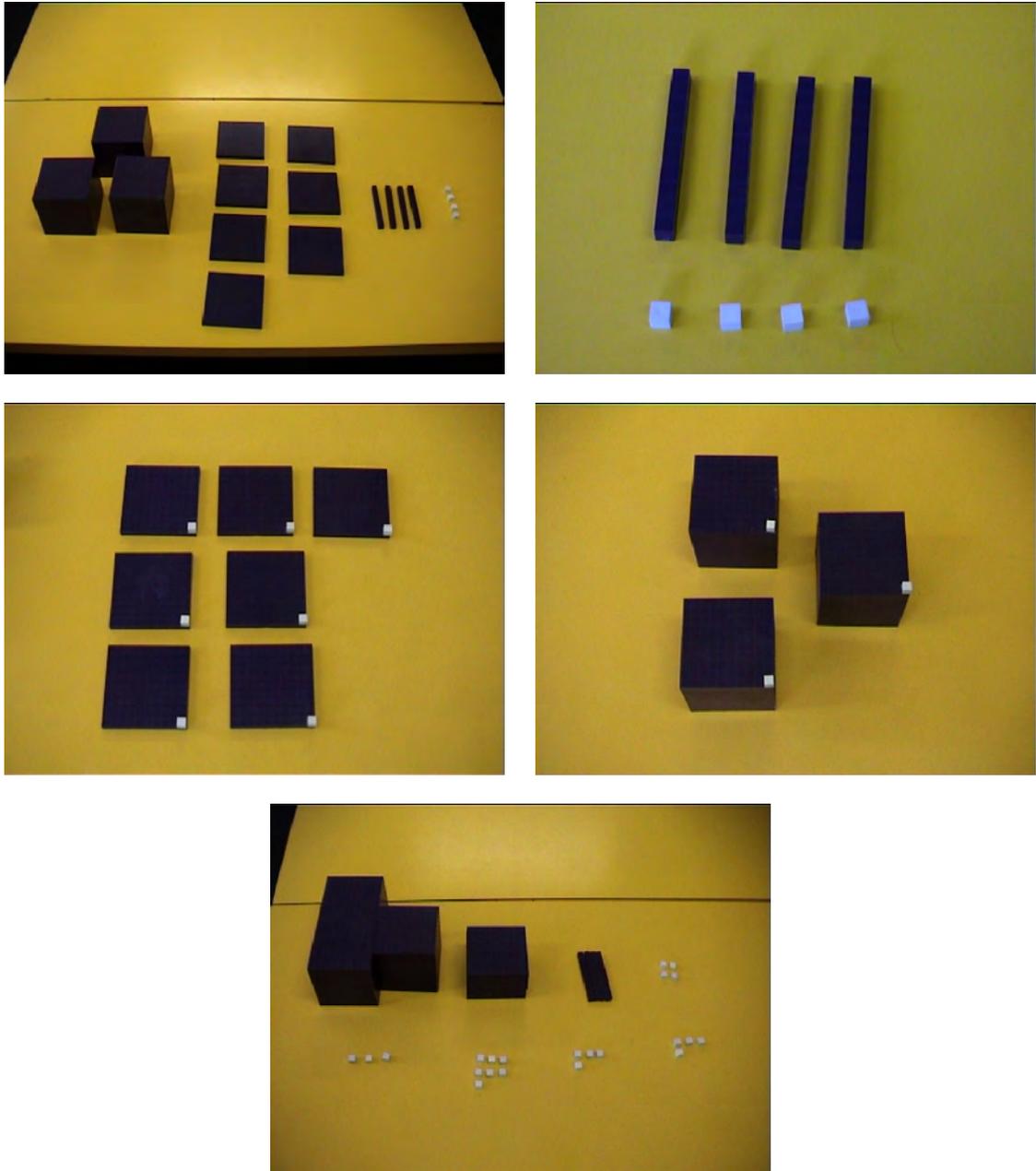


Figure 1. Using Dienes' blocks to demonstrate by measurement division that 3744 is divisible by 9.

This can be followed by the conventional discussion using the distributive property that is found in most text books.

$$\begin{aligned}
 3744 &= 3(1000) + 7(100) + 4(10) + 4 \\
 &= 3(999+1) + 7(99+1) + 4(9+1) + 4 \\
 &= 3(999)+3 + 7(99) + 7 + 4(9) + 4 + 4 \\
 &= [3(999) + 7(99) + 4(9)] + 3 + 7 + 4 + 4,
 \end{aligned}$$

where the square-bracketed term is divisible by nine.

The frustrating thing for some students is that although these methods show whether or not a number is divisible by nine, they do not produce the answer! This provides a motivating connection to the work on partition division.

A demonstration of short division using Dienes' blocks can be used to stress the difference between measurement division, as used in the divisibility by nine problem, and partition division, as used when the answer to a problem is the goal. This time the problem, $9)3744$, is used with nine sets laid out on the floor and connections made to place value and the written algorithm on the white board. This is shown in a few steps in Figure 2. First 30 hundreds are distributed with 3 hundreds left over. These are combined with the original 7 hundreds and redistributed, leaving 1 hundred. This is redefined as 10 tens and combined with 4 tens, to be distributed with 5 tens left over. Finally the tens are redefined as units and the 54 units are distributed, 6 to each set. The necessity to do multiplication and subtraction as part of the algorithm creates connections among all operations. Short division can be contrasted with long division, and the lack of necessity to model or produce an algorithm for the latter, if the former is understood, can be discussed (Watson, 1995). The use of the calculator and estimation are again stressed. What to do with remainders (when they occur) should also be addressed and connected to situations associated with measurement and partition division contexts.

Connections to prior experiences

Testing tea-bags

Rich mathematical activities can be developed from mundane objects. A trip to the supermarket finds a new kind of tea-bag with the claim that it has 50% more room for the tea leaves to move. The bag is pyramidal in shape. Students can be asked to work in small groups to assess the truth or otherwise of the claim, using other kinds of tea bags as comparisons, and to produce a poster-style report. This leads to immediate discussion about the phrase '50% more'. Does it refer to the surface area of the tea bag or the volume within? If the latter, how can it be measured? These and other similar questions emphasise the need to define terms and conditions before embarking on a mathematical investigation.

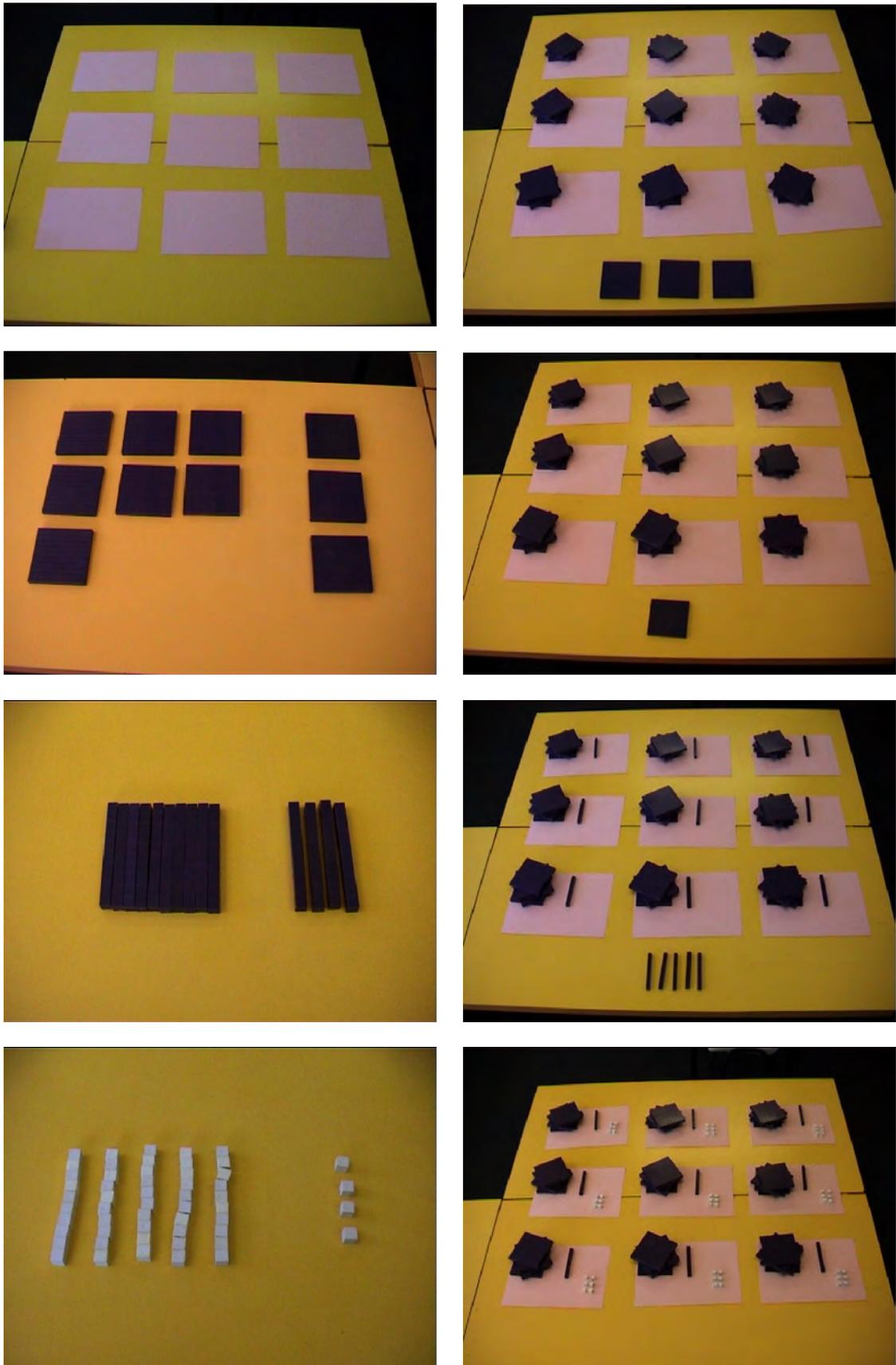


Figure 2. Steps with Dienes' blocks to demonstrate by partition that the answer when 3744 is divided by 9 is 416.

Students might investigate the claim in different ways. Some might choose to weigh the tea, others might consider the degree of coarseness. Taking tea bags apart leads to a discussion about surface area and volume. These activities provide opportunities for students to discuss their mathematics, refine their problem solving approaches and consider the ‘real world’ meaning of the claim. They also link to prior knowledge, so that a teacher might prompt discussion by referring to mathematics already encountered. Cross-curriculum aspects of science, social science, art, English, and technology could conceivably be included in work arising from this investigation, providing further opportunities to build connectedness into the mathematics program.

Complete the square

Filling in spaces on a 5 by 5 grid provides a more mathematical problem solving investigation. Students roll a die and fill in the gaps according to the number rolled. Posing the question ‘how many rolls will it take to fill the board’ provides a basis for another rich investigation. To answer this question students could do a number of things. Some might choose to do it experimentally with no prior planning. Questions such as ‘What do we know about board games?’ or ‘Is the same number likely each time?’ help students make connections to their intuitive understanding and experience about chance. Those who start by thinking about the limits — the smallest and largest number of rolls that it could take — are already making connections with number work and basic skills. What happens if four boards are placed together? Does this mean that the average number of rolls quadruples? Again this connects to other areas of mathematics, and provides a reason and stimulus for learning about averages and questioning results. What does 7.8 rolls mean anyway?

Students find that using familiar materials in unfamiliar ways not only is motivating, but also helps them to see mathematical ideas at work. It emphasises the connectedness of mathematics to other curriculum areas and to life outside the classroom, and particularly connects new ideas to their current mathematical and contextual understanding.

Relevance connections

Newspapers (or indeed the news media more generally) provide an opportunity to link literacy and numeracy at the middle school level and reinforce mathematical skills in relevant contexts.

The ‘Numeracy in the news’ website with *The Mercury* newspaper in Hobart (<http://ink.news.com.au/mercury/mathguys/mercury.htm>) provides many examples for teachers of ways in which they can connect the mathematics classroom with appropriate, and sometimes inappropriate, examples of numeracy. Students in the later years of middle schooling could be asked to find examples of newspaper articles where numeracy is important, discuss a usage of it in an article, and suggest the appropriateness for the situation of the way in which the numerical information is used.

It is surprising how easy it is to find appropriate material in newspapers, although some students may choose articles with too many numbers and become overwhelmed

with the task. Often the shortest articles with a few salient quantitative expressions provide the most fertile ground for developing ideas. When this type of activity has been used with pre-service teachers, there has been considerable enthusiasm expressed about the opportunities to develop cross-curriculum connections, particularly to science and social science.

Figure 3 contains a recent newspaper article that illustrates the potential for connections to be made across curriculum areas. It requires some regional geography skills to locate Fiordland and discuss the kind of terrain it is. It also requires some scientific understanding of gravity and the units used to measure it. Finally what does it mean to say ‘gravity in southwest Fiordland is 150 units above average’? This article could be the basis of some interesting project work.

Heaviest scenery on Earth

It doesn't matter how hard you've dieted, how much lettuce you've eaten, your weight's up in Fiordland.

Scientists have found that gravity is stronger in the southwest of the South Island than almost anywhere else on Earth.

That means people weigh more there than elsewhere.

The difference is too small to be picked up by the bathroom scales.

But it is a sign that the area — known for its scenic wonders — is in for some heavy geological activity within the next few million years.

A study published in the American

journal *Science* says Fiordland is one of the newest places in the world where Earth's plates are grinding together, creating a new zone of earthquakes and volcanoes similar to the Taupo volcanic zone.

This leaves heavy rocks from the planet's core relatively close to the surface, causing higher gravity.

The study says gravity between Doubtful Sound and Dusky Sound in southwest Fiordland is 150 units above average.

'It's one of the strongest positive gravity anomalies on the planet, over land,' said one of the study authors, Professor Peter Kamp of Waikato University.

Figure 3: *New Zealand Weekend Herald* 5–6 Oct, 2002, p1.

Conclusion

The examples of activities presented here are only a few of the many possibilities for creating connections in the middle school classroom. We have found that students at all levels of education become intrigued and motivated by these kinds of investigations. In conclusion, this quote from the father of problem solving, George Polya (1945, p. 15), seems to say it all:

One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else. We have a natural opportunity to investigate the connections of a problem when looking back at its solution.

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Challenging and effective teaching in junior primary mathematics: What does it look like?*

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Effective mathematics teachers make a difference, but what is it specifically that effective mathematics teachers do? In this article, we will describe a classroom research study of six effective Prep to Grade 2 teachers that may provide some pieces of the puzzle. This study took place within the Victorian Early Numeracy Research Project. Our belief is that these findings give a framework that can be used as the basis for reflection on good practice, at preservice and inservice levels.

Background

Given the success of particular Asian countries in the Third International Mathematics and Science Study (TIMSS), there was considerable interest in describing commonalities between classrooms of particularly effective countries. Much interest focussed on Japanese primary classrooms in particular. Through observation and videotape, Stigler and Stevenson (1991) conducted a study of 120 classrooms in Taipei (Taiwan), Sendai (Japan) and Minneapolis (USA). The role assumed by the teacher in the Asian countries was that of

knowledgeable guide, rather than that of prime dispenser of information and arbiter of what is correct. There is frequent verbal interaction in the classroom as the teacher attempts to stimulate students to produce, explain, and evaluate solutions to problems (p. 14).

Brown (1998, p. 2) noted that international observational studies seem to show some agreement on some of the aspects of teacher quality which correlated with attainment. These included:

* This paper has been subject to peer review.

- the use of higher order questions, statements and tasks which require thought rather than practice;
- emphasis on establishing, through dialogue, meanings and connections between different mathematical ideas and contexts;
- collaborative problem solving in class and small group settings; and
- more autonomy for students to develop and discuss their own methods and ideas.

In a major study of effective primary school mathematics teaching in the United Kingdom, Askew, Brown, Rhodes, Johnson and Wiliam (1997) studied the practices of a number of different teachers, with varying levels of effectiveness.

The teaching practices of the highly effective teachers

- connected different ideas of mathematics and different representations of each idea by means of a variety of words, symbols and diagrams;
- encouraged students to describe their methods and their reasoning, and used these descriptions as a way of developing understanding through establishing and emphasising connections;
- emphasised the importance of using whatever mental, written or electronic methods are most efficient for the problem at hand; and
- particularly emphasised the development of mental skills.

As will be evident, the findings of these studies relate well to the findings from the present study.

The context of our research

The study discussed in this paper took place as part of the Early Numeracy Research Project (ENRP) in Victoria, Australia, a collaborative project between the Department of Education and Training, the Catholic Education Office (Melbourne), the Association of Independent Schools Victoria, Australian Catholic University and Monash University. Three hundred and fifty-four P–2 teachers in thirty-five schools (approximately 240 per year) participated in a three-year research and professional development project, exploring the most effective approaches to the teaching of mathematics in the first three years of school. There were three key components of this project:

1. a research-based framework of ‘growth points’ in young children’s mathematical learning (in Number, Measurement and Space);
2. a 40-minute, one-on-one interview, used by all teachers with all children at the beginning and end of the school year;
3. extensive professional development at central, regional and school levels, for all teachers, coordinators, and principals.

Our first awareness of the term ‘growth points’ was in the work of Trish O’Toole and colleagues at the Catholic Education Office (Adelaide), and the first written use of the term, to our knowledge, was by Pengelly (1985). The framework of growth points will

not be discussed in detail here (for a fuller description, see Clarke, 2001). However, the intention was to describe the typical ‘learning trajectory’ of five- to eight-year olds.

There were four to six growth points in each mathematical domain. To illustrate the notion of a growth point, we will discuss the domain of Addition and Subtraction Strategies. Consider the child who is asked to find the total of two collections of objects. Many young children ‘count-all’ to find the total, even when they are aware of the number of objects in each group. Other children realise that by starting at one of the numbers, they can count on to find the total. Counting All and Counting On are therefore two important growth points in children’s developing understanding of addition.

A one-on-one, interactive, hands-on interview was then developed that could provide classroom teachers with rich information on what their children knew and could do (both individually and as a class) across a variety of domains, with particular insights into the strategies used in solving problems. The disadvantages of pen and paper tests have been well established by Clements (1995) and others, and these disadvantages are particularly evident with young children, where reading issues are of great significance. The interview has much to offer the teacher of young children, if time and resources permit. Further information on the interview can be found in Clarke (2001).

At the time of writing the interview had been used with over 36 000 children at P-4. For each interview, teachers completed a four-page record sheet. The information on these sheets was then coded by a trained team of coders, assigning achieved growth points to each child for each domain. This process, including statistical measures to convert the growth point data to an interval scale, is discussed in detail in Rowley and Horne (2000).

The initial professional development was intended to prepare teachers to use the interview. Over the remaining three years of the project, the professional development focus was on taking what was learned from the interview to inform planning and teaching for maximum effectiveness, both cognitive and affective. Schools formed *professional learning teams* for this purpose. An example of the power of professional learning teams at one school is reported in Clarke (2002).

Identifying particularly effective teachers

In identifying effective teachers, our interest was in growth in student understanding across the school year. Some children, for reasons of home background, language background or other factors, come to school with less mathematical understanding and skills than others. Our emphasis on growth, however, allowed us to choose teachers that make a difference for all children.

By aggregating data on children’s growth in terms of movement through the ENRP growth points, the first two years’ data was then used to identify particularly effective teachers — the ones whose children showed the greatest growth over two years, for each of two cohorts of children.



Using these data, we chose six teachers for case studies. There was one at Prep, one at Grade 1, one at Grade 2, and teachers of composite grades — a Grade P/1 and a Grade 1/2 teacher.

We also chose another Prep teacher who had made particularly impressive gains in a setting where almost all children were from non-English speaking backgrounds. The range of grades was in recognition that teaching Prep children mathematics is different in several ways to teaching Grade 2, for example.

Studying what effective teachers do

The six case study teachers were studied intensively through use of the following data sources:

- five lesson observations by two researchers, incorporating detailed observer field notes, photographs of lessons and collection of artefacts (e.g. worksheets, student work samples, lesson plans);
- teacher interviews following the lessons;
- teacher questionnaires completed through the duration of the project; and
- teacher responses to other relevant questions and tasks posed to them.

Teachers were observed by two researchers, working together for five lessons. Lessons on three consecutive days were observed in the middle of the school year, and then lessons on two consecutive days were observed a couple of months later. Teachers were asked to focus on different broad content, of their choosing, in each of the two sets of lessons (e.g. Number the first set, Space the second).

Both observers used laptop computers to take notes on the lesson, and teachers were interviewed after each lesson (interviews being audio-taped and transcribed), as they discussed their intentions for the lesson, and what transpired.

Many photographs were taken also, to give a richer picture of the classroom environment and the nature of the activities used. In all, eighty-six researcher visits were made to schools during the case studies, including intensive practice of the whole team at one school.

Decisions needed to be made on the kinds of notes that would be taken on the lessons. The decision was taken to attempt to note as much of possible of what transpired in the lesson in a relatively 'free' form. We did however agree on a broad framework for the observations and interviews (see Figure 1).

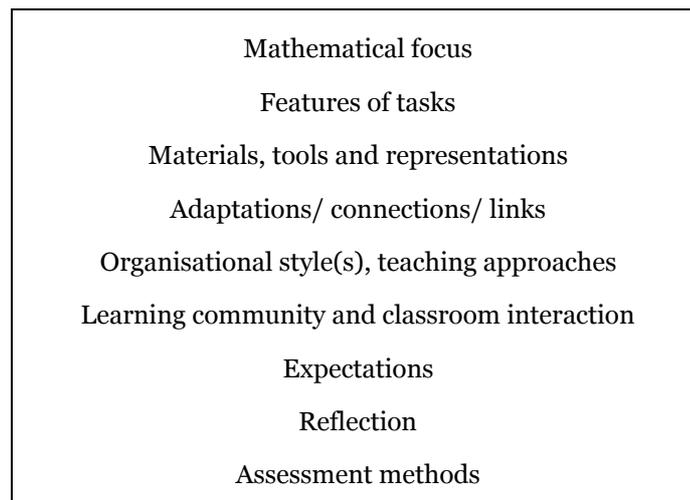


Figure 1. Categories within the ENRP lesson observation and analysis guide.

Our aim was to describe the practice of demonstrably effective teachers and to ultimately look for common themes, not to judge.

Having two observers in each classroom was an important feature of the methodology. Of course, two different people saw different things on occasions, and the reflection together after each lesson provided an opportunity to share these insights, and to develop a plan for subsequent observations and interviews.

At various stages during the case study, the team met together to describe to each other what they were seeing. 'Critical friends', not involved in the research, provided feedback on the kinds of themes they were hearing, as verbal reports were made. The first three lessons and the subsequent discussion prompted the team to focus on particular aspects in the last two lessons and interviews that had not necessarily been noted to that point, but which had emerged in team discussions.

What can be said about effective teachers?

Following the lesson observations, interviews, analysis and a number of research team meetings, it was decided to use the original framework to describe the practices of effective teachers. It was agreed to list common elements where evidence was available for at least four of the six teachers. The description of common themes or features of what effective teachers do, as revealed in this study, is given in Figure 2.

Effective teachers of P–2 mathematics...	
Mathematical focus	<ul style="list-style-type: none"> • focus on important mathematical ideas • make the mathematical focus clear to the children
Features of tasks	<ul style="list-style-type: none"> • structure purposeful tasks that enable different possibilities, strategies and products to emerge • choose tasks that engage children and maintain involvement
Materials, tools and representations	<ul style="list-style-type: none"> • use a range of materials/representations/contexts for the same concept
Adaptations/connections/ links	<ul style="list-style-type: none"> • use teachable moments as they occur • make connections to mathematical ideas from previous lessons or experiences
Organisational style(s), teaching approaches	<ul style="list-style-type: none"> • engage and focus children’s mathematical thinking through an introductory, whole group activity • choose from a variety of individual and group structures and teacher roles within the major part of the lesson
Learning community and classroom interaction	<ul style="list-style-type: none"> • use a range of question types to probe and challenge children’s thinking and reasoning • hold back from telling children everything • encourage children to explain their mathematical thinking/ideas • encourage children to listen and evaluate others’ mathematical thinking/ideas, and help with methods and understanding • listen attentively to individual children • build on children’s mathematical ideas and strategies
Expectations	<ul style="list-style-type: none"> • have high but realistic mathematical expectations of all children • promote and value effort, persistence and concentration
Reflection	<ul style="list-style-type: none"> • draw out key mathematical ideas during and/or towards the end of the lesson • after the lesson, reflect on children’s responses and learning, together with activities and lesson content
Assessment methods	<ul style="list-style-type: none"> • collect data by observation and/or listening to children, taking notes as appropriate • use a variety of assessment methods • modify planning as a result of assessment
Personal attributes of the teacher	<ul style="list-style-type: none"> • believe that mathematics learning can and should be enjoyable • are confident in their own knowledge of mathematics at the level they are teaching • show pride and pleasure in individuals’ success

Figure 2. Common themes emerging from ENRP case studies of effective teachers.

The six teachers were chosen from 150 who were involved in the first two years of the project on the basis of the growth in understanding and skills over two years of teaching, using an instrument in which we can have great confidence. The one-on-one interview provided far richer and more important data than can possibly be revealed by standardised testing, particularly in Grades P–2. That is, we can be confident that the teachers chosen as effective were done so on the basis of student improvement in the things that the mathematics education community generally values.

It has been interesting since this list has become public, to have the opportunity to discuss it with other ENRP teachers than those involved in the case studies. Many ENRP and non-ENRP teachers have found the list affirming of either their current practice or the practice to which they aspire. The list also finds support from the research studies cited earlier in this paper.

Although only six teachers were studied intensively, our visits to the classrooms of other project teachers (578 school visits in all), led us to believe that the features in Figure 2 were increasingly evident over the three years of the project in other classrooms, as teachers took what they had learned from the interviews about children's mathematical thinking, and, working with colleagues, endeavoured to provide the kinds of activities and tasks that enhanced learning for all students. In a questionnaire at the end of the project, all teachers were asked to identify the greatest changes in their teaching practice. Among the most common were: more open-ended tasks and activities; more probing questioning/ asking why and how/ valuing children's thinking; challenging and extending children/ higher expectations; more practical/ hands-on activities; and greater emphasis on reflection/ sharing (see Clarke, Cheeseman, Gervasoni, Gronn, Horne, McDonough, Montgomery, Roche, Sullivan, Clarke, & Rowley, 2002).

Snapshots from the classrooms of effective P-2 teachers

Sometimes, a list such as that in Figure 2 can seem a bit removed from the life and colour of the classroom. In order to give a picture of vibrant kinds of learning communities we observed during the study, we now discuss some classroom examples that illustrate some of the themes evident in the table.

Effective teachers of mathematics encourage children to explain their mathematical thinking/ideas and build on children's mathematical ideas and strategies

One of the benefits of the regular use of the interview was that the kinds of tasks used and questions posed provided models of the kinds of tasks and questions that could be used in classrooms. Teachers commented that they found themselves using many more questions that probed children's thinking. Examples included:

- How did you work that out?
- Could you do that another way?
- How are these two objects the same, and how are they different?
- What happens if I change this here?
- What could you do next?
- Can you see a pattern in what you've found?
- Can you make up a new task using the same materials?

Effective teachers of mathematics focus on important mathematical ideas and make the mathematical focus clear to the children

Teachers noticed during the interview that although many children could read and write two- and three-digit numbers, quite a few had difficulty ordering one-digit numbers.

In school teams, they developed a range of games and activities that focussed on this important mathematical idea.

One teacher asked children to cut up magazines and catalogues, taking out any numbers they could find. They then sorted these out, in order from smallest to largest. The two examples (Figure 3) give a sense of range of children's responses.



Figure 3

Another teacher developed a card game, where the picture cards were removed from a standard pack, and children had half the pack each. At the same time, each child turns over a card, and the person with the larger of the two numbers takes both. Once again, there was a clear focus on ordering numbers from smaller to larger.



Figure 4

In conclusion

It is interesting to consider the extent to which the list of twenty-five teacher behaviours and characteristics in Figure 2 has application to other grade bands. We believe that similar research in Grades 4–12 (and possibly beyond), would yield many elements in common with these. Indeed, a discussion of this would be most worthwhile among preservice and inservice teachers.

It was a wonderful privilege to be in the classrooms of dedicated, mathematics education professionals. We have attempted to describe their practice in ways that ‘ring bells’ for readers. We have described classrooms where the enthusiasm, curiosity and strategies of young children are valued and built upon, with lasting effects upon their understanding, their attitudes, their love of mathematics, and their confident views of themselves as learners of mathematics.

Acknowledgements

We wish to acknowledge the insights reflected in this paper of our colleagues in the research team (Jill Cheeseman, Donna Gronn, Ann Gervasoni, Marj Horne, Andrea McDonough, Anne Roche, Glenn Rowley and Peter Sullivan), Pam Hammond from the Department of Education, Employment and Training (Victoria), and our co-researchers in ENRP schools.

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Mathematics activities used with students at risk in Year 9

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All of the activities described in this workshop have been used with Year 9 students who are at risk either academically and/or socially. The activities are not all original but have been collected from a number of places over the years. During the workshop each activity will be 'played' and an explanation of why it was used and how it was accepted by students will be discussed.

The activities include: *Date Maths*, *Maths Bingo*, *Step Maths*, *The Great Aussie Horse Race*. These activities require little preparation and/or planning and can usually be run on the spot especially if the normal class work is likely to deteriorate.

Introduction

This workshop illustrates some of the activities that I have used with Year 9 STAR (STudents At Risk) classes. These students have been removed from the mainstream classes because of problems stemming from anti social behaviour induced by either learning difficulties or inappropriate social skills. The class size was kept below 22, but on any given day the attendance would range from 10 to 15 with a different mix of students every time. The following activities are simple and easy to use. I have only five or six activities that I repeat at random every class time. It has been found that by repeating activities students become accustomed to them and so promotes better participation and when the rules of the activity are familiar they can use their mathematics skills instead of concentrating on the rules. At the beginning of the year I usually introduce one new activity each week.

Date Maths

Equipment required: Nil

Time: 10+ minutes

Select a student from the class and ask for their birth date, e.g. 15/9/92

Ask the students of the class to write the numbers from 1 to 10 down the left hand side of their page, as shown below. Also write it on the board.

- 0 =
- 1 =
- 2 =
- 3 =
- 4 =
- 5 =
- 6 =
- 7 =
- 8 =
- 9 =
- 10 =

Using only the numbers that appear in this date to make mathematical expressions that equal the numbers from 0 to 10.

After about 10 minutes (or some other suitable time — depends on the attention span of students) ask different students to provide the answers. You will have helped some students and know that they have a correct answer, so involving them in supplying the answers makes them belong to the activity. Keep the hardest expressions for the brighter students. Often there is more than one way to write the statement (as indicated in Table 1).

Table 1. Typical answers for Maths Date activity for the date 15/9/92

Date = 15/9/92

0	= $(9 - 9) \times 152$	$(5 \times 2 - 1 - 9) \times 9$	
1	= $1 + (9 - 9) \times 52$	$5 - 2 - 1 - 9/9$	
2	= $2 + (9 - 9) \times 15$	$95 - 92 - 1$	$95 - 91 - 2$
3	= $1 + 2 + (9 - 9) \times 5$	$19 - 9 - 2 - 5$	$9/9 + 5 - 2 - 1$
4	= $5 - 1 + (9 - 9) \times 2$	$95 - 92 + 1$	
5	= $5 + (9 - 9) \times 12$	$29 - 19 - 5$	$9/9 + 5 - 2 + 1$
6	= $5 + 1 + (9 - 9) \times 2$	$95 - 91 + 2$	$(5 + 2 - 1) \times 9/9$
7	= $5 + 2 + (9 - 9) \times 1$	$19 - 9 + 2 - 5$ $9 - 5 + 9/(1 + 2)$	$9/9 - 1 + 5 + 2$
8	= $(1 + 5 + 2) \times 9/9$	$9/9 \times 1 + 5 + 2$	
9	= $9/9 + 1 + 5 + 2$	$9 - 9 + 5 \times 2 - 1$	$9 + 9 - (5 \times 2) + 1$
10	= $5 \times 2 + (9 - 9) \times 1$	$(99 + 1)/(5 \times 2)$	

Discussion

Asking the students to provide the date at the beginning of the activity lets them know that the teacher has not contrived the numbers in some special way before class. Also the teacher becomes an active participant in finding the statements.

In the above example the power of using 'zero' is demonstrated (9–9). It takes some students time to realise that a number multiplied by zero always gives zero. Of course a particular date might not have two of the same numbers to make an easy zero, and this can make it challenging for both the students and the teacher. The students like to see me struggle to find an answer and this empowers some of the brighter students to attempt the problems and solve it before me. On really difficult problems I often give them an incentive to find the answer by the next morning and I will give them some reward (a drink at the canteen is quite good enough).

For the slower students I will put some mathematical statements on the board and ask them to work out which numbers they match.

So what is the benefit of doing this activity? It makes students think about number operations and order of operations. Often in class they are given a number problem and asked to find the unique answer. But here they are given an answer and they are asked to generate a non-unique problem. This activity enhances their number skills and basic number properties. This activity is a cognitive challenge and I definitely do not use it more than once a fortnight

Do all students enjoy and participate in this activity? For student in the STAR class the answer is No! A good number of STAR students are anti instructions anyway so I do not expect them all to jump joyously into the activity, but as the year progresses I often see them working out some of the problems or interjecting when they think I have written something wrong. Often their minds are doing some of the activity even if they are not writing it down or appear to be actively participating. When I have used this activity with mainstream classes the majority of students are prepared to participate and be challenged.

Maths Bingo

Equipment required: 12 sided die numbered 1 to 12 (or a pack of shuffled cards).

Time: 10+ mins

Ask the students to draw a grid of squares 5 wide and 5 deep as shown below.

The teacher (or student) rolls the dice and the students are told the number. The students are to write this number in any square in the grid. Repeat this until all squares are full.

The students are to try and group the numbers so that they have all the same numbers together or as a run of consecutive numbers, in rows or columns.

Scoring: 5 numbers the same or run = 100
 4 numbers the same or run = 50
 3 numbers the same or run = 20
 2 numbers the same or run = 10

Aim: To get the highest score.

Student to write totals for each row at the end of the row and the total for each column at the bottom of each column. Then add all these totals for the grand total.

Figure 1 shows the grid that students would draw and figure 2 provides an example of a completed table.

Figure 1. The grid that students draw

1	4	5	6	3	10
1	7	7	7	2	20
1	1	1	1	1	100
2	6	8	9	2	10
9	10	5	4	3	30
30	0	0	10	40	240

Grand total

Figure 2. An example of a completed table

Discussion

The scoring system above is only a suggestion. When students are familiar with the activity I will ask different students to suggest scores for each section. They will often want scores like 27, 43 etc. until some of the quicker students will object that those numbers would make it hard to add up.

Students get an appreciation of probability. A whole game can be played where one or two numbers may never be rolled. That surprises some of them and I sometimes enhance the expectation by saying something like ‘since we haven’t had a 6 yet it should come very soon’.

The term ‘consecutive numbers’ becomes part of the students’ vocabulary and they can see them forward, backward, down and up. The tension mounts as the last few squares are filled in, the students want very particular numbers to complete their patterns. Some become so good that they can place numbers in such a way that they obtain scores vertically and horizontally to their advantage.

When the squares are filled then the whole room becomes a buzz of students adding up their scores. For 2 or 3 minutes I can actually relax and watch mathematics take control of the class. However, at the beginning of the year when the activity is first introduced, a lot of help needs to be given in showing students how to obtain their scores.

Step Maths

Step Maths

Equipment: 10 sided die numbered from 0 to 9.

Time: 10+ mins

Ask the students to draw the steps that are shown figure 3.

Roll the die and call the number. This number must be put in the top box of the steps

Roll the die, call the number and this number can be put in one of the two boxes in the next step down. On the next roll the number must be put in the remaining box of the second step.

Repeat this for each step until you get to the bottom and all boxes are full of numbers

Students are now to add the numbers in each column as in ordinary addition. Thus add the numbers in the right hand column, if the total is greater than 9 write down the units number and carry the tens to the next column.

Aim: To obtain the greatest total.

Figure 3 shows the steps that the students draw and Figure 4 provides the first four steps of a typical problem with the total by adding up the columns.

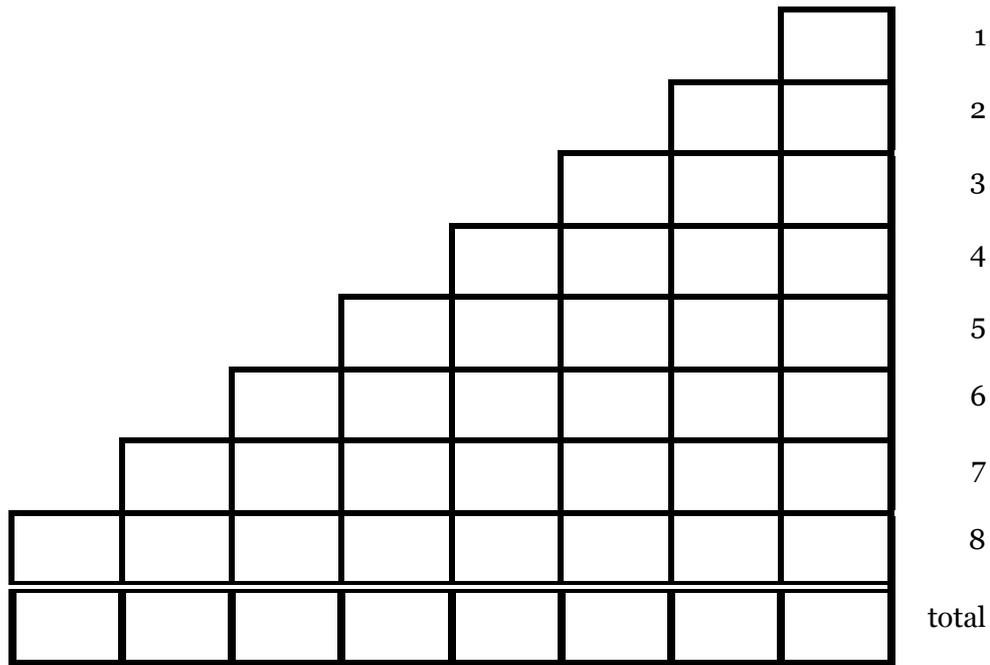


Figure 3. The steps that the students draw for Step Maths.

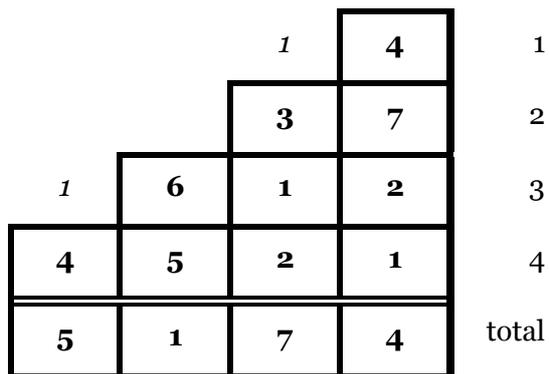


Figure 4. A completed game for four steps

The total for the sample in Figure 4 is five thousand one hundred and seventy four = 5174. I ask the student to carry the next place value to the top of the next column, it helps when you need to check totals that are in dispute.

Discussion

The most obvious learning outcome of this activity is the importance of place value. It might take students a few games before their strategies become refined and to weigh the probabilities up as to where to place a number such as to maximise their score.

You will also find students in the class that can not tell the difference between steps that go up or down from left to right. An eye needs to be kept on them when they are drawing the steps and help given to draw the steps the correct way.

The totals will be in the order of 100 million. As the students give you the totals get them to say the number correctly. This will take some effort for a lot of students.

To make things different the aim can be to find the least total. Or reverse the steps so that they go down from left to right. If you are short of time make fewer steps.

The Great Aussie Horse Race

Equipment: 2 × 6 sided dice, (prepared board)

Time: Allow at least 30 minutes to complete this activity.

Have on the board the grid shown below (A grid could be prepared for each student to fill out.)

Ask students to select a number from 0 to 13. Place their names against the numbers.

The sum of the roll of the two dice will be used to select which horse will get to move.

Teacher rolls the dice first. The horse selected moves into the first space. Give the dice to the student with that horse. That student rolls to select the next horse. Repeat until a horse reaches the finish line (or until 5 minutes before the end of class).

The grid on the board with a partly entered horse race is shown in Figure 5.

The Great (name of school) Horse Race

The race

H o r s e N u m b e r s		Name of horse																	
	0	John																	
	1	Mary																	
	2	Allan	X																
	3	Bill	X	X	X														
	4	Moses	X	X	X														
	5	Henry																	
	6	Judith	X	X	X	X	X	X	X										
	7	Micheal	X	X	X														
	8	Joe	X	X															
	9	Cindy	X																
	10	Werty																	
	11	Hassem																	
	12	Ducky	X																
13	Ful																		

Figure 5. A partly completed grid for the horse race activity.

Discussion

This activity is a good one to use when you are teaching probability distributions or graphing. I might only use this activity once a term as it can take a whole period and the students can get quite excited with the whole procedure.

Once the 'horses' have been assigned a number and the students see that you have two dice some will make the connection that number 1 and 13 will never get a go. No many students will realise that the numbers 6, 7, 8 should occur more times than the other numbers. As the graph starts to grow the students slowly become aware that it is not an even distribution and they will start 'backing' the horses that are likely to win. Every time that I have used this activity a different distribution occurs with some unusual surprises.

Rewards

Most games have a winner and they expect some reward. I try and make more than one winner with as many people as possible receiving rewards. So what rewards do I use? I have found that writing the students' names on a special area on the board so that they can see their name is all that is required. If there is some special task or duty that needs to be done I will select students from this list

How to maximise the number of rewards: For date maths I write the maths statement, if it is correct, on the board with the students name next to it. I choose carefully who will provide the answers so that as many students as possible can have a turn.

For Maths Bingo and Step Maths the first person to get a total has that total and name written on the board, then the next total that is greater than this is written on the board, this is repeated until we get the highest total for the class is found. By the time students have checked each others work 8 to 10 students names might be on the board.

Conclusion

These are not the only activities that can be used so look out for other good ones that are easy to implement and fun to do, but with some mathematical learning involved.

Word of caution, do not overplay an activity. Always finish an activity with the students demanding to do it again. In this way you will know that when you do it next time it will be welcomed with anticipation.

Integrating notebook computer technology into Year 10 mathematics

Sean Ebert

Penrhos College, WA

The challenge has been to integrate the use of the computer into the classroom in a realistic manner to enhance understanding and increase levels of technological skills. Over the past four years our Mathematics Department has developed many different computer-based activities that relate directly to our girls' needs. This presentation will demonstrate some of our activities and give ideas on how other teachers could integrate technology to enhance learning. The presentation has applications to teachers in schools with notebook programs or those using computer labs.

Penrhos College has had an extensive notebook program in place for the past five years. Every student from Years 5–10 is expected to own a notebook computer and have it at school with them every day of the school year. In simple terms, there are approximately 700 students with a notebook computer at our school at any one time, an exciting yet daunting prospect. The logistics behind keeping such a large number of computers (at least another 100 staff notebooks) in working order is amazing. Every room in the school has access to the Internet via connection points along the walls. The information age is truly upon us.



Every student from Year 5 to 10 owns a notebook computer.

Against this backdrop, the Penrhos College Mathematics Department has developed a range of activities designed to enhance the learning experience of the students with the school. Use of the computers takes on a variety of forms. Some staff members use the computers as the principle method of teaching all concepts. Generally, however, staff members favour a balanced approach of the old and the new, incorporating the technology where appropriate in order to maximise results.

This is the approach that has been used in our Level One Year 10 Mathematics course. The two Level One classes usually contain about 50 of the more capable students representing the top one third of the girls in the year group. An added consideration is the fact that our Year 11 and 12 tertiary entrance subjects require the use of a graphics calculator. We have addressed this issue by loading a computer version of the HP39G onto each of the student's notebooks so that the girls are not disadvantaged with respect to students at other schools who use the graphics calculators extensively in Year's 8, 9 and 10.

The first topics that we cover are Space and Measurement including volume and surface area. Following is an activity that the students complete after three lessons of formal teaching covering the basic concepts and formulae.

Spreadsheet activity: Surface area/Volume

Part 1

Using Microsoft *Excel*, set up a spreadsheet using the following headings:

Length	Surface Area	Volume	Length/Length1	SA/SA1	Volume/Volume1
1	=6*A2^2		=A2/\$A\$2		

Use a length of 1 cm for the first row of your table. *Length1*, *SA1* and *Volume1* refer to the values obtained in this first row.

Use the formulae to find the surface area and volume of a cube and fill down to complete the table for lengths from 1 to 20 cm.

Comment on the relationship between the ratios found in the last 3 columns of your table.

Suggest what happens to the surface area and volume of cube if the length is changed by a factor of n .

Convert 1 m to cm.

Convert 1 m² to cm².

Convert 1 m³ to cm³.

Explain how these conversions are related to your earlier findings.

Part 2

Using Microsoft *Excel*, set up a spreadsheet using the following headings:

Radius	Surface Area	Volume	Radius/Radius ₁	SA/SA ₁	Volume/Volume ₁
1					

Use a radius of 1cm for the first row of your table. *Radius₁*, *SA₁* and *Volume₁* refer to the values obtained in this first row.

Use the formulae to find the surface area and volume of a sphere and fill down to complete the table for radii from 1 to 20cm.

Comment on the relationship between the ratios found in the last 3 columns of your table.

Suggest what happens to the surface area and volume of sphere if the radius is changed by a factor of n .

Part 3

Use the same method to find the relationship for different 3-dimensional figures e.g. cones, cylinders etc.

This is a good example of the style of the computer activities used in our course. The technology needs to assist in the development of understanding, not simply be an irrelevant addition.

The next activity shows how several styles of learning can be linked together in order to produce a lesson that introduces a new and quite difficult concept in a meaningful context. This activity works particularly well as it requires not only computer skills, but also some construction techniques.

Spreadsheet activity: Latitude

At the end of this activity you should have a clearer understanding of how to find the distance around the Earth at different lines of latitude.

Part 1

The lines of latitude decrease in size as they move towards the two poles. The Equator has a radius of 6400 km and hence a circumference of approximately 40192 km. Draw a semicircle with a radius of 10 cm to represent a side view of the Earth as shown below:

Draw a line from the centre of the semicircle to the edge of the arc at an angle of 10° from the Equator. Measure the radius of the semicircle at this point (the distance y on the diagram).

Construct a table in *Excel* as shown below. Continue to do this for angles of 10° , 20° , 30° and so on to 90° .

Angle x	Radius y	Circumference	Ratio	Cosine
0	10	62.83185307		
10				
20				
30				
40				

$=2 \cdot \pi \cdot B$ $=C/2$ $=\cos(A \cdot \pi / 180)$

Part 2

Plot your values of angle against radius below:

Part 3

Use Graphmatica to graph the equation $y = \cos(x)$.

Change the graph paper in View to Trig.

Your graph in Part 2 should look like the graph of $y = \cos(x)$ up to $x = \pi/2$.

Part 4

The circumference of any of the lines of latitude are found by the formula:

Length =

How does this formula relate to the trigonometric ratio cosine?

Prime Numbers an activity that has evolved over time. The first time the students attempted this activity it was designed to be an application of quadratic functions. Initially, as shown, the students were directed to a website and then expected to download the first 1000 prime numbers. This proved quite a task with many students unable to connect at once. On reflection, the Internet search was unnecessary and superfluous to the desired result of the activity. In subsequent years the girls have been given the list of prime numbers.

Internet/ Spreadsheet activity: Prime numbers

Part 1

Use the Internet to find a list of the first 1000 prime numbers. Try the Search Engine Yahoo and type in *prime numbers* or go to www.utm.edu/research/primes. Keep this list as a word document to help you with Part 2.

Find the rule for determining Mersenne primes.

Part 2

Use a spreadsheet to generate prime numbers as shown on page 319 of your textbook. Complete Questions 1 to 7. Find when each of the generators breaks down by extending your table and using your list of prime numbers.

x	$2x^2 + 11$	$x^2 + x + 11$
1	$=2 \cdot 1^2 + 11$	$=1^2 + 1 + 11$
$=1^2 + 1 + 11$	Fill Down	Fill Down
Fill Down		

The use of laptop technology is only effective if the students can apply the concepts learned by using them. The *Simultaneous Equations* activity has been the most successful application of this expectation. It combines a number of different methods of solving simultaneous equations including algebraically, graphically and trial and error.

Computer activity: Simultaneous equations

The staff at Sohrnep Agricultural Farm has a problem. They are required to count the stock at the end of each week so that the feed for the following week can be ordered. The farm has sheep (4 legs) and ducks (2 legs).

Unfortunately the staff are very lazy and to make their job easier they only count legs and heads.

At the end of one week they count 50 heads and 124 legs. How many sheep and ducks are there?

Part 1

Write two equations using the information above.

$$s + d = \underline{\hspace{2cm}}$$

$$\underline{\hspace{1cm}}s + \underline{\hspace{1cm}}d = 124$$

Solve the two equations simultaneously.

Part 2

Open Graphmatica and graph the two equations you have used above. Remember to use x and y instead of s and d.

You will need to go to **View** and Grid Range and change Top to 50 and Right to 50.

Go to **Point** and Coord Cursor to check your answer in Part 1.

Part 3

Open a spreadsheet. Fill in the cells as shown below.

	A	B	C
1	Sheep	Ducks	Legs
2	50	0	200
3	49	1	198
4	48	2	196
5	47	3	194
6	46	4	192
7	45	5	190
8	44	6	188
9	43	7	186

=A2-1 =\$A\$2-A3 =A2*4+B2*2

Once you have done this, click on the cells with formulae in them and drag down by clicking and holding down on the black square in the bottom right corner. When column C returns an answer of 124 you have gone far enough.

Explain in your own words why cell A2 has the number 50 in it.

Explain why column C is going down by 2 each time.

The students completed the simultaneous equations activity as an investigation following classroom instruction on solving algebraically. The visual aspect of *Graphmatica* coupled with the trial and error method of a spreadsheet helped consolidate understanding. The activity brought together the best aspects of both the teacher centred and student centred learning models. The trick is to design all computer activities to work as well as this one has and thereby use the technology to enhance learning within the classroom.

Tests without marks: A criteria-based approach to providing students with meaningful feedback

Vince Geiger, Robyn McNamara & Pauline Vallely

Hillbrook Anglican School, Qld

Recent syllabus changes in Queensland have encouraged schools to consider the implementation of approaches to assessment that are genuinely criteria based. The teachers at Hillbrook Anglican School have developed an assessment program that aims to provide the means of making judgements about student performance using written descriptors of standards within each criteria rather than more traditional approaches such as marks. This session will describe the Year 11 mathematics assessment programs introduced in 2002 and provide opportunity to discuss the challenges, advantages and opportunities faced and experienced by staff and students during the development and implementation of these programs.

Background

Secondary schools in Queensland have operated under a school based, externally moderated assessment regime for the best part of twenty-five years. Further, for most of this time, assessment practices have been governed by criteria based rather than norm based approaches. While such an approach does not prohibit the use of marks when determining students' summary grades, recent syllabus change has provided the opportunity to take advantage of programs that are more outcomes based in flavour than has traditionally been the case. Specifically, these changes now determine that the grades students receive upon completion of a course must be consistent with standards descriptors within three criteria prescribed by the Queensland Studies Authority in each of the Mathematics syllabuses under its imprimatur. It is arguable that consistent alignment with these standards and those implemented at school level can best be achieved through task specific standards descriptors of student performance. This paper is an attempt to describe how one school, Hillbrook Anglican, developed such an assessment approach.

Reasons for taking a criteria-based approach

Hillbrook's assessment standards had for some time made use of verbal descriptors of student performance in some aspects of work — mostly associated with assignment tasks. Given the nature of the new syllabus initiatives, and a sense that there were potential advantages by way of student feedback and reporting to parents, it was decided to attempt to develop an assessment program based around a broad range of tasks, both test and assignment based, in which standards of student performance would be assigned by matching students work to verbal descriptors and not marks.

The major advantages that were perceived to be available were:

- establishing clear and unambiguous standards — each assessment item is accompanied by a task specific criteria sheet that provides a clear indication of what is required to meet each standard of performance within each of three criteria;
- making the process of assessment more transparent — students are provided with task specific criteria sheets well in advance of the due date of an assessment item with the intent of making it very clear what is required for success on any item;
- providing clear and unambiguous feedback to students and parents — having established what was required for achieving particular standards, advice on how why a certain standard was not reached and what areas need to be targeted for improvement could readily be provided by reference to the task descriptors and the student's work.

Initial implementation of the program suggests that each of these advantages has been realised to at least a moderate level of success.

Development of test criteria

The first test of this type was developed as a team effort and involved all members of the Mathematics Department teaching at a senior level. Initially, a test was written by one member of the team and was compiled on the basis of developing a 'good test' based on the teacher's experience with the subject. The associated criteria sheet was also developed by the teacher, again referencing their experience and to answer the questions, 'What should a C student be able to do?' and 'What should a B student be able to do?' etc. This initial draft was presented to the other members of the team for scrutiny and improvement. Particular care was taken in considering whether the developing standards were appropriately demanding for students at each level of achievement. This began a cyclic process of development and feedback until all members of the team were comfortable with the product.

This process also acted as a model for the development of tests in the other two mathematics subjects offered by the school in the senior secondary program.

Implementation

Students were provided with the criteria and standards for the test they were about to sit about a week before their exam block. Time was taken to explain the process to the students and they were encouraged to ask any questions for clarification. Initially some students appeared a little sceptical that this was ‘as good as using marks’, on the other hand there were comments such as, ‘Is this all I need to do to get a C?’ which indicated that the process was as informative as had been hoped.

Students sat the exam under standard test conditions.

Some unexpected advantages

A difficulty arose in the development of one of the tests. The same process was followed as outlined earlier in that a ‘good test’ was compiled and then an attempt was made to develop suitable standards. The teacher who had taken major responsibility in this case, however, found that they could not write descriptors for an ‘A’ student in one of the criteria. The initial concern was that perhaps the approach we were taking was not appropriate to the particular subject. Upon further reflection with the participating teachers, however, it was determined that the test itself did not contain any questions that were of sufficient demand to allow students to demonstrate their capacities at the required level. Once a number of such questions were added descriptors were relatively easy to write. In this way the need to write appropriate standards informed the quality of the test instrument with the end result being an instrument that allowed students the opportunity to show what they knew and could do at a range of achievement levels.

A second anecdote reinforces the benefits of the transparent nature of this program. Upon the return of one the tests to students after grading, the teacher also provided a clean copy of the associated standard descriptors. Students were invited to scrutinise the teachers work but were only to question their grade when they could show they had met a descriptor where the teacher had judged they had not done so. Teachers would be aware that the return of papers usually initiates an avalanche of inquiries and requests such as ‘Can I have a half mark for this?’ – in this case there was not one question.

Last comments on a work in progress

The initial impression, from both staff and students, is that this is an effective process for determining levels of student achievement. I think members of the teaching team would admit that the development of this type of task and associated standard descriptors is a labour intensive process. The advantages which follow from this hard work, however, are the development of more targeted instruments that are potentially more informative for students, teachers and parents alike. The success of this program will only become apparent after its use through the two year assessment program in which this paper describes the development of one item – but progress to date has been encouraging. A sample of one of the test items has been attached for your consideration and comment.

Question 1

Simplify the following, expressing powers with positive indices:

a) $(3x^2)^3$

b) $\sqrt{a} \leftrightarrow a^{\frac{3}{2}}$

c) $\frac{a^2c^4 \leftrightarrow a^5c^{-6}}{a^3c}$

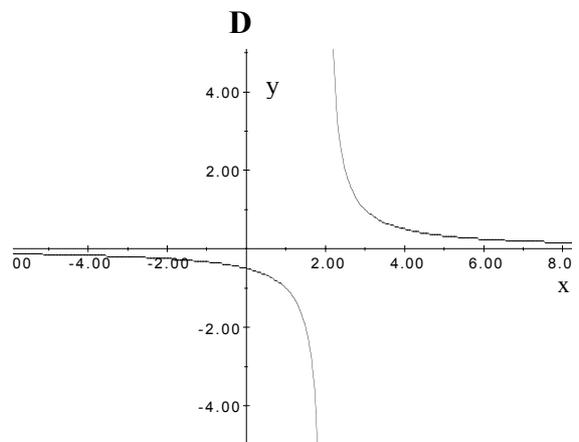
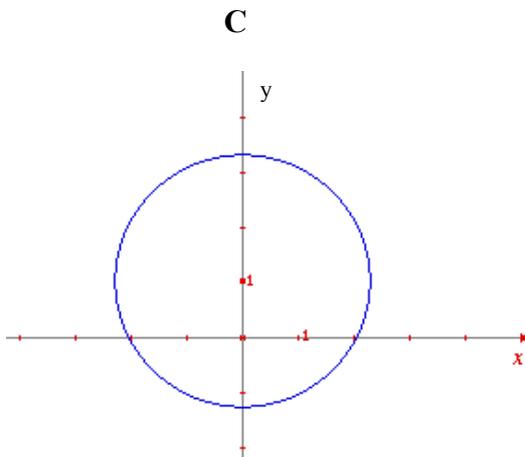
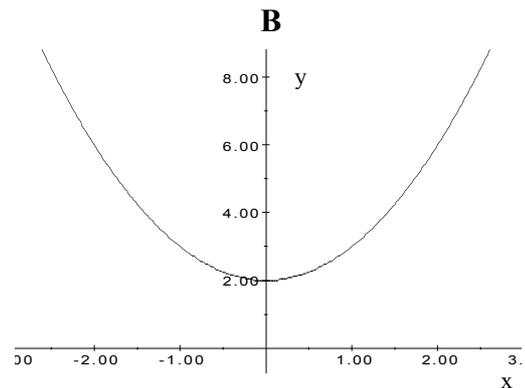
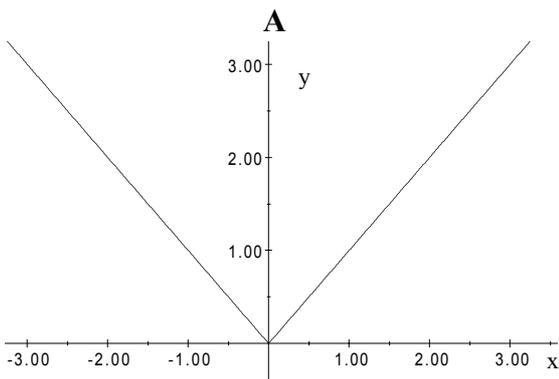
d) $\frac{3^n \leftrightarrow 9^{n-1}}{81}$ (express answer as a power of 3)

Question 2

Use the four graphs below to answer the following questions. You should consider each graph to be a **smooth** curve/line. State answers clearly by referring to the graph names as Graph A, Graph B, Graph C or Graph D.

a) Which of the graphs represent a **function**? Give a reason for how you determined your answer(s).

b) If the graph represents a function, state its domain.



Question 3*

Each of the situations, referred to below, can be modelled by a member of a particular family of functions. Name the **function family** that **best** represents each of the activities or phenomena listed below. Your response should include the name and a quick sketch (axes must be ruled and labelled but scale markings are not required) of each function.

- a) The cost of a taxi ride with a \$3.80 flag-fall (minimum charge starting fee) and then \$1.30 per kilometre thereafter.
- b) A dying star loses one third of its brightness every year.

Question 4

a) Given the function $f(x) = (x + 2)^2 - 5$ find:

i) $f(3)$

ii) $f(x+1) - f(x)$

b) Find the inverse function, $f^{-1}(x)$, of $f(x) = (x + 4)^3$

c) Use your calculator to find the points of intersection between $y = 2x + 1$ and $y = -x^2 + 4$. Show the keystrokes you used in this process.

Question 5*

Grant has set up a company to manufacture toy cars. He has fixed costs of \$40 per week. These costs are due to rent on his premises, registrations and insurances that are charged whether he sells any goods or not. The cost of materials to produce a single car is \$10. Grant decides that he will sell the cars to the public at \$20 each.

a) Use the above data to copy and complete the following table showing profit per week in relation to number of cars sold per week.

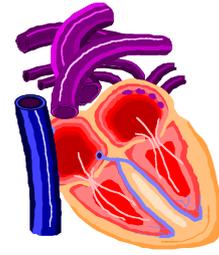
Number of Cars sold per week (n)	0	1	2	3	4	5
Profit in dollars per week (p)						

b) Use the data in the table to find the function relating profit to number of cars sold.

Question 6***

The blood pressure created by the heart's contraction varies and for a particular patient can be modelled by the periodic piecewise function shown below. This means the pattern repeats itself every second.

When the left ventricle contracts it delivers blood to the body with considerable force and blood pressure is raised to about 150 millimetres of mercury. Between contractions the pressure is lowered to about 80 millimetres of mercury. So blood pressure (P), in mm of mercury, in the heart changes with time (t), in seconds.



$$P = \begin{cases} 80 + 700t - 1700t^2 & 0 \leq t \leq 0.3 \\ 162 - 82t & 0.3 \leq t \leq 1 \end{cases}$$

Typically blood pressure is taken after the patient has been resting for approx. 60 seconds. Your task is to find the difference between the maximum and minimum blood pressure for this patient when the maximum blood pressure occurs at time 60.21 seconds and the minimum blood pressure occurs at 61.0 seconds.

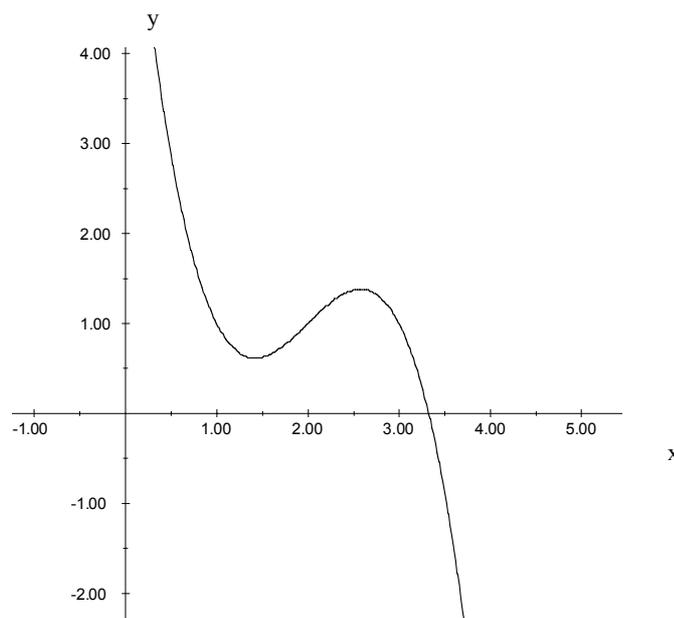
Question 7

State precisely in a sentence (or in point form) how the original functions listed on the left below have been transformed to make the final functions. A sketch is optional.

Original Function	Final Function
a) x^3	$3x^3$
b) $ x $	$- x $
c) x^2	$(x + 3)^2 - 4$

Question 8 **

Find the equation of the graph below. Explain clearly how you determined this equation. Ensure that you show the basic function that the graph is based on and list any transformations that have occurred.



Question 9*

The amount, A , of a financial investment grows (increases) by 2 % every month. The amount invested at the beginning was \$2,500.

- a) Copy and complete the table below for the first two months of growth. Show your working.

Time, t (months)	0	1	2
Amount, A (\$)			

- b) Find the equation or model that relates the Amount (A) to the Time (t).
 c) Find the final amount of this investment after 1.5 years.

Question 10 ***

When designing a car, manufacturers are ever more careful about how they design safety features of their products, for example, airbags, brakes etc. In order to improve such equipment car makers are constantly designing and testing new ideas. This question is related to a manufacturer's tests on a new braking system.

The **total stopping distance** of a car is related to two factors:

1. The *Reaction Distance*. This is the distance a car travels while the driver reacts to a situation and then applies the brakes.
2. The *Braking Distance*. This is the distance a car takes to stop once the brakes are applied.

The Trektron car company has designed a new braking system and needs to know how effective it will be under normal driving conditions. A test is conducted for a variety of speeds. The results of these tests are displayed below:

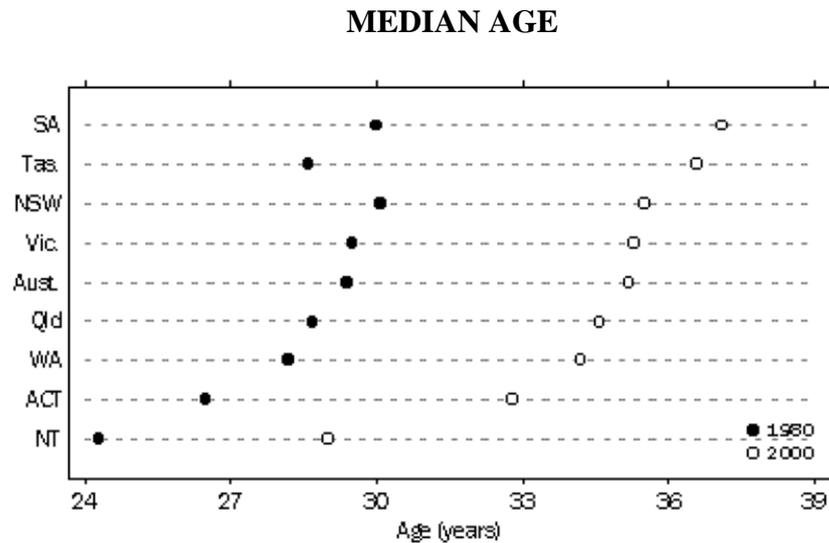
Reaction and Braking Distances		
Speed (km/hr)	Reaction Distance (m)	Braking Distance (m)
32	6.096	11.198
48	9.144	22.52
64	12.192	35.268
80	15.24	49.427
96	18.288	62.169
112	21.336	74.676

This is useful information once interpreted but it doesn't tell us the total stopping distance of a car when it is moving at the types of speeds people have to use on roads. What the manufacturer wants to know is the total distance that a car will take to stop, once the driver is given a signal to do so, when a car is travelling at 60 km/hr. The manufacturer hires you to find out.

Make use of a mathematical model(s) to determine the **total stopping distance** for a car that is travelling at 60 km/hr.

Question 11

The graph below is from the Australian Bureau of Statistics website and it relates to the population of Australian citizens. It shows the breakdown by States and Territories of the median age of citizens for the years 1980 and 2000.



- From the graph, what was the median age of NSW citizens in 2000? (A reasonable approximation will be sufficient.)
- What percentage of the NSW population is older than this age?
- Write one (1) valid conclusion based on a trend that can be drawn from the data in the graph. Support your conclusion with data from the graph.

Question 12 *

The **data** provided on the **next page** (after Question 13) contains information about tropical rainforests in selected countries. You need to use this data for this question. This question is concerned with comparing original rainforest area compared to the known area in 1990.

- What percentage of the original total area was left in 1990?
- Use your calculator to draw comparative box plots of the Original rainforest area and the Present (1990) rainforest area. Sketch these plots neatly on your lined paper, indicating clearly the five number summary for each plot.
- Comment on the differences between the two plots. Your comments should include discussion of the median, range and relative spread of each distribution. How does the middle 50% of each distribution compare?
- Which country was no longer an outlier in 1990?
- Make a list in your calculator showing the differences between the Original rainforest area and the Present (1990) rainforest area. Use this to find the mean loss of rainforest over this period.
- Which country had lost the largest total area of rainforest by 1990?

Question 13 **

Two archers, Steve and Gavin, are competing against each other in a special competition. They play a total of 11 rounds and the maximum score in any one round is 25. At the end of the competition, there is some disagreement between observers as to who is the best archer. The results obtained in each round for each archer are displayed in the table below.

Scores in each Round (maximum score = 25 in each round)											
Steve	21	22	21	19	18	23	21	19	20	18	22
Gavin	21	24	17	18	20	21	23	21	25	17	20

Using appropriate statistical and/or graphical techniques, analyse the data to clearly justify who **you** believe is the best archer.

END OF PAPER

The following data is related **Question 12**.
This is the data that was given to you before the exam.

Knowledge and procedures

CRITERIA SHEET – END SEM. ONE

	A	B	C	D	E
<p>Objectives</p> <ul style="list-style-type: none"> recall definitions and results access and apply rules and techniques demonstrate number and spatial sense demonstrate algebraic facility demonstrate an ability to select and use appropriate technology such as calculators, measuring instruments and tables demonstrate an ability to use graphing calculators and/or computers with selected software in working mathematically select and use appropriate mathematical procedures work accurately and manipulate formulae recognise that some tasks may be broken up into smaller components transfer and apply mathematical procedures to similar situations. 	<p>The paper <i>consistently</i> demonstrates the:</p> <ul style="list-style-type: none"> accurate recall, selection and use of definitions, results and rules appropriate use of technology appropriate selection, and accurate and <i>proficient</i> use of procedures. <p>Indicators for this grade include the capacity to:</p> <p>As for B plus:</p> <ul style="list-style-type: none"> Draw valid statistical comparisons based on the analysis of data and through the use of sophisticated statistical measures (inter-quartile range, standard deviation etc.) or graphical displays Determine the inverse of a function Make use of piecewise functions for modeling and/or making predictions with data Identify the original function and the transformations used to produce a graph <p>There may be a small number of minor errors or omissions in one or two of the <i>specific</i> areas of stated.</p>	<p>The paper <i>generally</i> demonstrates the:</p> <ul style="list-style-type: none"> accurate recall, selection and use of definitions, results and rules appropriate use of technology appropriate selection and <i>accurate</i> use of procedures. <p>Indicators for this grade include the capacity to:</p> <p>As for C plus:</p> <ul style="list-style-type: none"> Draw valid statistical comparisons based on the analysis of data and through the use of simple statistical measures or graphical displays Develop simple linear models based on a set of paired data Make use of functional notation Make a connection between a life related situation and a function that provides the basis for a reasonable model Identify the original function and transformations used to produce a graph Set up and find some of the parameters for determining a exponential model for a set of paired data points Use mathematical models to make predictions <p>There may be a major error/emission or a small number of minor errors or omissions in one or two of the <i>specific</i> areas of stated above.</p>	<p>The paper <i>satisfactorily</i> demonstrates the:</p> <ul style="list-style-type: none"> accurate recall and use of <i>basic</i> definitions, results, rules and procedures appropriate use of <i>some</i> technology accurate use of <i>basic</i> procedures. <p>Indicators for this grade include the capacity to:</p> <ul style="list-style-type: none"> Find simple statistical measures such as percentage Find simple measures of central tendency Perform simple interpretations of graphical representations Obtain the value of a function given a value of x. Perform simple simplifications involving indices Determine if a relation is a function Identify the procedures for the simple transformation of functions Find domain and range of different functions Use technology to find points of intersection <p>There may be a small number of minor errors or omissions in one or two of the <i>specific</i> areas stated above or one major error/omission.</p>	<p>The paper <i>occasionally</i> demonstrates the:</p> <ul style="list-style-type: none"> accurate recall and use of definitions, results and rules appropriate use of <i>some</i> technology. <p>Indicators for this grade include the capacity to:</p> <p>As for C :</p> <p>There may be a number of major errors or omissions in <i>specific</i> areas stated.</p>	<p>The paper <i>rarely</i> demonstrates any knowledge about or use of procedures.</p> <p>Chosen mathematical techniques are <i>generally inappropriate</i>.</p>

Modelling and problem solving

	A	B	C	D	E
<p><i>Modelling</i></p> <ul style="list-style-type: none"> understanding that a mathematical model is a mathematical representation of a situation identifying variables of a simple mathematical model of a situation forming a mathematical model of a life-related situation deriving results from consideration of the mathematical model chosen for a particular situation interpreting results from the mathematical model in terms of the given situation 	<p>The overall quality of a student's achievement across the full range within each context, and across topics, generally demonstrates high quality mathematical thinking.</p> <p>Indicators include:</p> <p>As for B plus:</p> <ul style="list-style-type: none"> Identifying essential features of the triple asterisk questions. Developing valid models and/or solutions for the triple asterisk questions. A capacity to: <ul style="list-style-type: none"> explore the strengths and limitations of the models where appropriate. understand a problem or task in a deep and insightful way. <p>It is likely that a student who demonstrates these abilities will solve a significant number of single, double and triple asterisk questions.</p>	<p>The overall quality of a students achievement across a range within each context, and across topics, generally demonstrates characteristics of good mathematical thinking.</p> <p>Indicators include:</p> <p>As for C plus:</p> <ul style="list-style-type: none"> Identifying essential features of the double asterisk questions. Substantial progress is made toward a valid model and/or solution is developed for the given single asterisk questions and double asterisk questions. a capacity to: <ul style="list-style-type: none"> interpret, clarify and analyse conditions and variables select and use effective strategies for at least the double asterisk select appropriate procedures for a problem or task synthesise procedures and strategies. <p>It is likely that a student who demonstrates these abilities will solve a significant number of single and double asterisk questions.</p>	<p>The overall quality of a students achievement in all contexts generally demonstrates some characteristics of mathematical thinking.</p> <p>Indicators include:</p> <ul style="list-style-type: none"> Identifying essential features of single asterisk questions. Appropriate strategies and/or procedures have been identified for at least single asterisk questions. Some progress has been made toward the development of a model/solution for at least the single asterisk questions. <p>Students may demonstrate these skills on any of the asterisked questions but the minimum requirement is that they are displayed on single asterisked problems.</p>	<p>The overall quality of a students achievement, across contexts and topics, sometimes demonstrates elements of mathematical thinking.</p> <p>Indicators include:</p> <ul style="list-style-type: none"> using basic procedures appropriately using appropriate basic strategies <p>Student's have at least made a genuine attempt at single asterisk questions but may not have made much progress toward developing a model/problem solution.</p>	<p>The overall quality of a students achievement, across a range within each context, and across topics, rarely demonstrates mathematical thinking.</p> <p>There is no apparent progress toward developing models or finding solutions to any of the asterisked problems.</p>
<p><i>Problem solving</i></p> <ul style="list-style-type: none"> interpreting, clarifying and analysing a problem using a range of problem-solving strategies such as estimating, identifying patterns, guessing and checking, working backwards, using diagrams, considering similar problems and organising data understanding that there may be more than one way to solve a problem selecting appropriate mathematical procedures required to solve a problem developing a solution consistent with the problem developing procedures in problem solving. 					
<p><i>Investigation</i></p> <ul style="list-style-type: none"> exploring a problem and from emerging patterns creating conjectures or theories selecting and using problem-solving strategies to test and validate any conjectures or theories extending and generalising from problems 					

Communication and justification

<p><i>Communication</i></p> <ul style="list-style-type: none"> organising and presenting information communicating ideas, information and results appropriately using mathematical terms and symbols accurately and appropriately using accepted spelling, punctuation and grammar in written communication understanding material presented in a variety of forms such as oral, written, symbolic, pictorial and graphical translating material from one form to another when appropriate presenting material for different audiences in a variety of forms (such as oral, written, symbolic, pictorial and graphical) recognising necessary distinctions in the meanings of words and phrases according to whether they are used in a mathematical or non-mathematical situation. 	<p>A</p> <p>The student <i>consistently</i> communicates information <i>clearly, effectively and efficiently</i>.</p>	<p>B</p> <p>The student <i>generally</i> communicates information <i>clearly and effectively</i>.</p>	<p>C</p> <p>The student <i>generally</i> communicates information at a <i>satisfactory</i> level. There may be a number of instances of untidiness or illegibility. Language may have been used loosely on a number of occasions, making sections difficult to interpret. Exam may be poorly organised or there may be minor communication omissions and/or errors.</p>	<p>D</p> <p>Some terminology and symbols are used correctly or in accordance with convention. Presentation may be poor or inappropriate format may have been used. Little meaning can be obtained from the exam because of the loose use of language.</p>	<p>E</p> <p>The <i>overall quality</i> of a student's achievement <i>rarely</i> demonstrates use of the basic conventions of language or mathematics.</p>
<p><i>Justification</i></p> <ul style="list-style-type: none"> developing logical arguments expressed in everyday language, mathematical language or a combination of both, as required, to support conclusions, results and/or propositions evaluating the validity of arguments designed to convince others of the truth of propositions justifying procedures used recognising when and why derived results are clearly improbable or unreasonable recognising that one counter example is sufficient to disprove a generalisation recognising the effect of assumptions on the conclusions that can be reached deciding whether it is valid to use a general result in a specific case using supporting arguments, when appropriate, to justify results obtained by calculator or computer 	<p>Conclusions and results used by the student are logically argued and they provide evidence of complex reasoning.</p> <p>This includes:</p> <ul style="list-style-type: none"> Justification of procedures Thorough verification of models and solutions Presenting valid and well argued conclusions. 	<p>Conclusions and results used by the student are supported through the use of mathematical reasoning to develop simple logical arguments and results.</p> <p>This includes:</p> <ul style="list-style-type: none"> Justification of procedures Some verification of models Presenting a valid and logically argued conclusion. 	<p>The student makes use of some mathematical reasoning to develop simple logical arguments in support of conclusions and results.</p>	<p>Results and conclusions used by the student are rarely supported by justification.</p>	<p>Results and conclusions used by the student are not supported by justification.</p>

Online learning: Using the Internet to explore mathematics

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Course participants will explore the various applications in a hands-on manner, with computer and Internet access.

This workshop will focus on mathematics education in the secondary school. Course participants will be led through the six categories of Internet applications that will assist student learning as well as teacher preparation. These categories encompass lower to higher order thinking, covering a range of mathematical abilities, focussing on Bloom's taxonomy and Vygotsky as well as incorporating instructivist and constructivist applications.

In addition, participants will develop strategies to find appropriate Internet resources for classroom teaching, improving their own Internet search skills.

With the advent of various computer-based technologies, mathematics education is being transformed to dynamically support the classroom theory students learn, extending students' understanding of mathematical applications and experiences in the world, through the medium of the World Wide Web.

Learning mathematics — the past and the future

Although the principles cannot change, with regard to the future study of mathematics, it could be said that, 'the facts and procedures typically mastered in mathematics curricula are the 'vocabulary' of mathematics, the tools of the trade, which, like all tools are only meaningful when they are used' (Schoenfeld, 1988, p. 69).

It could be argued that regardless of the historical period, mathematics has served its purpose for the people living at that time. If they were alive today, Archimedes, Euler and Newton would still understand the mathematics of today (with some small adjustments). Essentially, it is not the subject that has changed, but the means by which understanding can be gained (Schoenfeld, 1988, p. 68), using various technologies.

Fundamentally, students of mathematics, just like mathematicians of the past, still need 'to see what makes things tick, and how they are internally and externally connected', 'to abstract and perceive uniformities in objects and systems that are

superficially different' and experience 'abstraction, formal systems, and other mathematical methods of inquiry as fundamental means of investigating these structures and relationships' (Schoenfeld, 1988, pp. 68–9)

Thus, the role of mathematics education is an evolving one. In recent times, technology-based learning has begun to play a more significant role in the way mathematics is taught in schools. In addition, Pea (1987) suggests that we increasingly use 'cognitive technologies as reorganisers rather than amplifiers of mind' (p. 89) as the workforce requires workers to be multi-skilled, adaptable to change as well as flexible. As a result, there has been a shift in the premise of what role schooling takes in preparing students, 'from *what* to learn to *how* to learn' (Nickerson, 1988, p. 295). Learning need to be a lifelong practice because 'technology... also affects the content of education, because among the objectives of education is that of making understandable the world in which one lives, and we live in a technological world' (Nickerson, 1988, p. 285).

In the mathematics curriculum, technological applications enrich students' abilities to reason mathematically during the compulsory schooldays and beyond. Educators must incorporate meaningful technological opportunities into tasks, to ensure cognitive development occurs. Garofalo et al. (1998) suggests that 'the use of technology in mathematics teaching is not for the purpose of teaching about technology, but for the purpose of enhancing mathematics teaching and learning with technology'. In particular, using the Internet as a learning tool, educators can encourage students to develop a dynamic, visual understanding of mathematical concepts.

Pedagogical practices using the Internet

To successfully utilise the Internet and integrate technology into the mathematics curriculum, teachers need to make informed decisions on how to provide meaningful learning experiences. Consider the learning and teaching theories of Bloom and Vygotsky.

Bloom's taxonomy divides the cognitive domain into six areas:

- Knowledge
- Comprehension
- Application
- Analysis
- Synthesis
- Evaluation (Woolfolk, 1993, p. 443)

Vygotsky suggests that humans possess two distinct mental functions, lower and higher mental functions, whereby students' lower mental functions are inherited and involuntary, whereas their higher mental functions are learned and only exhibited voluntarily (Passey & Samways, 1997, p. 5).

The knowledge and comprehension aspects of Bloom's taxonomy are considered to be within the realm of lower-order thinking. As we move towards synthesis and

evaluation, in the context of Vygotsky's ideas regarding mental functions, these require the higher order thinking skills. The Internet is a valuable tool, in that it provides opportunities to stimulate students' mental functions at all levels.

As educators, we can use the Internet within the framework of these learning strategies, to facilitate learning for students, utilising both instructivist and constructivist learning theories (Grabe & Grabe, 2001, p. 119). This means that students can practice the skills and knowledge components of learning mathematics using the Internet, and, in addition, they can also explore, analyse and evaluate mathematical concepts, extending and enriching their knowledge and understanding.

Thus, there are six key educational components in computer-based learning:

- Educational Games
- Drill and practice
- Tutorials
- Exploratory data/ Investigations
- Simulations (Grabe & Grabe, 2001, p. 120–134)

I have added a further component, that is useful to the teacher:

- worksheets and teaching resources.

Educational games

It is essentially important for students to be enthusiastic and involved in activities that promote fun and reward success, even if they are extrinsic forms of motivation. With young people enjoying video and computer games, schoolwork has some real competition in gaining the attention of students. Games tend to have time limit incentives and depend on student motivation to ensure their success (Grabe & Grabe, 1998, p. 97). Generally, such applications are two-fold. Students need to answer the mathematics questions for a game to progress — thus sustaining their interest alongside understanding the game itself.

Examples of websites

- <http://www.quantumbrainbenders.com>
Problem solving.
- <http://www.coolmath.com>
Click 'ages 13-100'.
Click 'games'.
Under the heading 'thinking games', click 'Math word search'.
An interactive crossword should appear on the screen (there are about 6 different versions of the crossword).
- <http://www.superkids.com/aweb/tools/logic>
Interactive games — connect the dots(boxes), pegs (solitaire), towers of Hanoi.

Drill and practice

Similar to the idea of physically using flash cards (Grabe & Grabe, 1998, p. 95) drill-and-practice activities offer students the option of working at an individual pace. Tasks in this format have a limited function. They do not enable all students to understand (Grabe & Grabe, 1998, p. 99) and only motivate those who encounter success. The sites below, provide students with opportunities to revise skills.

Examples of websites

- <http://id.mind.net/~zona/mmts/trigonometryRealms/trigonometryRealms.html>
- <http://www.bun.falkenberg.se/gymnasium/amnen/matte/trigapplets/>
Trigonometric mathplets – trig functions: click ‘test your skills’. Java applet of radian measure.
- <http://janus.astro.umd.edu/astro/scinote.html>
Choose ‘let me try’ for students to practice scientific notation questions, answers checked by clicking a button.
- <http://www.bun.falkenberg.se/gymnasium/amnen/matte/trigapplets/>
Tutorial for practising conversions; degrees, radians etc.

Tutorials

Tutorials ‘present information and guide learning’ (Grabe & Grabe, 2001, p. 120). There are many of these available on the Internet, particularly through university websites.

Examples of websites

- <http://www.coolmath.com>
- <http://alepho.clarku.edu/~djoyce/java/Geometry/Geometry.html>
- <http://www.cecm.sfu.ca/pi/pi.html>
- <http://www.math.unc.edu/Faculty/mcombs/math22/growth&decay/keyexponentialgrowth&decay.html>
Homer Simpson and Flintstones questions.

Exploratory data/ investigations

‘Exploratory environments provide manageable and responsive... worlds for students to explore and manipulate’ (Grabe & Grabe, 2001, p. 134).

Vosniadeu and De Corte (1994) cite Hebestreit (1987) in suggesting that the most distinctive features of exploratory environments are their subjects, concrete-abstract objects. *Concrete* because in a sense they are real and *abstract* because they also exist within mathematical theory (p. 27).

The principles of the exploratory environment encompass ‘active learning... anchored in realistic situations, experiences, and goals’ (Grabe & Grabe, 1998, p. 104). They demand higher mental functions from students.

Research suggests that exploratory environments are not conducive to ensuring students encounter the learning experiences they need, just by giving them a topic to explore (Grabe & Grabe, 1998, p. 103–4). It can be better practice to provide students with an explicit outline/ structure or scaffold (Schoenfeld, 1988, p.75) including sample websites, point form of important information to find, etc., thus ensuring productive learning takes place.

Example of websites

- <http://www.unc.edu/~rowlett/units/index.html>
Units of Measurement website includes unusual units of measure. Prepare a summary of the unit chosen, include history — e.g. origin of the word, conversions (particularly with metric units if possible), where it is used — all on overheads, *PowerPoint*, cardboard.
- <http://www.ex.ac.uk/cimt/resource/resource.htm>
 Centre for Innovative Maths Teaching — Running the mile and high jump results.
- <http://pandora.nla.gov.au/nph-arch/2000/S2000-Jun-22/http://www.centenary.gov.au/>
 Centenary of Federation — timeline
- <http://www.teachervision.com/lesson-plans/lesson-5779.html?rn>
- http://www.teachervision.com/tv/tvsearch/lowest_grade=106&highest_grade=108&termname=Mathematics&type=Lesson%20Plan
 Investigations, ideas for lessons
- <http://www.shodor.org/interactivate/activities/perimeter/index.html>
 Shape Explorer
- <http://www.bom.gov.au/climate>
 Good for statistics. Click ‘climate averages and extremes’ then view options under heading ‘climate maps’.
 For example: Sunshine hours. Click on a city to view column graphs of different cities compare sunlight based on different months of the year.
 Other examples such as
http://www.bom.gov.au/climate/ahead/rain_ahead.shtml — predicting chance of exceeding median rainfall in the next 3 months in Australia.
- <http://www.census.gov.au/schools>
 Click ‘teaching materials’ then either ‘history’, ‘middle years’ or ‘later years’ for lessons, and information about the census.
 A number of census crosswords — key terms in stats etc. under ‘teaching materials’ then ‘quick activities’.
 ‘Student zone’ is also useful, more hands-on for students
- <http://www-history.mcs.st-and.ac.uk/history/>
 Mathematician of the day, history etc.

- <http://www.abs.gov.au>
Click 'education' then click 'teaching tools and classroom activities' explore any of the options on the page by clicking on the headings — lesson plans, worksheets, data etc.
- <http://www.cecm.sfu.ca/pi/pi.html>
The pi pages.

Simulations

Simulations 'provide controlled learning environments that replicate key elements of real-world environments' (Grabe & Grabe, 2001, p. 122). The main advantage of simulation-based activities, is their focus on developing and understanding of information, as opposed to memorising rules and formulas. Grabe and Grabe (1998). They provide opportunities for higher-order learning to occur.

There are many websites containing interactive java applets that can be manipulated to demonstrate a wide variety of mathematical concepts. There are also websites that combine tutorials with java applets, giving written explanations in conjunction with the applets. Java applets are appealing, in that students can manipulate the applet to experience the mathematics involved.

- <http://www.ies.co.jp/math/java>
Java applets
- <http://www.waldomaths.com/>
Java applets
- <http://www.hazelwood.k12.mo.us/~grichert/sciweb/mechanic.htm>
Projectile motion
- <http://www.shodor.org/interactivate/activities/imagetool/index.html>
Measuring distances, angles etc.
- <http://archives.math.utk.edu/visual.calculus/4/trapezoid.2/index.html>
Trapezoidal Rule
- <http://archives.math.utk.edu/visual.calculus/4/simpson.4/index.html>
Simpson's Rule
- <http://archives.math.utk.edu/visual.calculus/4/index.html>
Integration
- <http://archives.math.utk.edu/visual.calculus/4/areas.3/index.html>
The parabola

Teaching resources/ worksheets

- <http://puzzlemaker.school.discovery.com>
Choose Wordsearch, Math Square to start with.
- <http://www.edhelper.com>
Generates puzzles: word find, crossword, math crossnumber etc.

- <http://www.mathbuilder.com>
Generates individual worksheets (a unique code identifies them) — from Grades 4 to 10 (American site). Answers can be sent to your email address.
- <http://www.funbrain.com>
Tutorial — provides instant feedback to students.
Click ‘Ages’ (choose an age) — activities mainly focus on algebra, fractions, geometry (area and perimeter), naming co-ordinates, decimals
- <http://www.aplusmath.com/flashcards/index.html>
Many flashcards to choose from (java and non-java). Students can interact with the flashcards created and solve problems. Instant feedback is given (right or wrong + correct working).

Conclusion

In conclusion, there are many opportunities for students to utilise the Internet within the mathematics classroom. This technological approach, allows Mathematics to be experienced as a dynamic subject.

This paper also demonstrates that using the Internet is a valid educational tool. Learning can occur at multiple levels, for students of all abilities, from practice of basic skills and knowledge, to higher order exploratory and simulated mathematical experiences. Thus, allowing mathematicians of the future to explore and appreciate this field as much as the mathematicians of earlier centuries.

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A test for mathematics teachers

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What is mathematics? This was a question from one of my students and it is now the first in a series of questions that are used in this workshop to stimulate discussion. During the session it is intended that participants will gather a range of examples of mathematical concepts, processes and models that can be used for lesson planning. Also we will investigate some of the relationships between mathematical techniques, learning theories and business organisational models.

Each day mathematics teachers face the real test of defining and introducing mathematical concepts to their students. But as a first step how do we describe mathematics to our students and ourselves. There seems no shortage of attempts as every dictionary has a different definition and below is the definition from *Encarta* as an example.

Mathematics, study of relationships among quantities, magnitudes, and properties and of logical operations by which unknown quantities, magnitudes, and properties may be deduced (Microsoft Encarta Reference Library, 2002).

All of the definitions I have heard and seen have been (to say the least) thought provoking. Hence 'define mathematics' and 'define learning' are the first and last questions on the part one of the workshop handout. The title 'A Test for Mathematics Teachers' of course, is just a play on words and the content is a list of reflective questions designed to generate discussion.

For the purposes of this paper and workshop I have chosen concepts, processes and models as the basis of each part and I endeavour to provide and develop a series of alternative organised approaches to mathematics curriculum delivery.

The analysis of concepts

Most mathematical concepts are precisely defined compared to the concepts of social sciences. For example, one person's idea of love may differ from another. Also mathematics concepts are usually complex as compared to common concepts such as a chair. Students can feel a chair. They are sitting on one. A chair is real. But try to define an angle as is required in the workshop handout part one question five.

On one hand, an angle can be described as an action (as in rotational angle), which could be compared to kicking a football. However we seem to always draw angles as stationary objects. The original question in my first draft read, 'Is a point between the rays of an angle, part of the angle, or are the points on the rays the only parts of an angle?'; the question was left out because I felt it was too long and distractive.

Regardless of the complexity we need to understand the mathematical concepts we teach, and in my experience, this is learned on the job by sharing information between teachers. The workshop handout is designed to help you start discussions. However as teachers I am sure we can design our own questions.

Once we understand a concept we need to introduce it to the students. For example, I often use the following list to remind me to include the pure mathematics, the applications and the more people-focussed, investigative components in a program on a particular mathematical topic (Lane, 2000).

1. Develop the concept.
2. Establish the notation.
3. Define the operations.
4. Practice applications.
5. Solve problems.
6. Investigate the history.
7. Identify people's uses.
8. Design models.

The text books seem to be getting better and better at this aspect of mathematics and I advocate the use of a good textbook to help organise students thoughts in the understanding of complex mathematical concepts like angles. However the introduction of mathematical processes and models is more difficult because in different contexts, a range of alternative approaches can be implemented.

Process management

The analysis of these mathematical processes has helped to maintain my interest in this subject over many years, as a hobby, as well as a career, and this emphasis on process management has been a key difference between mathematics and other subjects. We have the responsibility to develop understanding of fundamental but abstract ideas such as angles and often use several approaches, from theoretical to tactile, to achieve this understanding.

The use of several lesson types in a multifaceted approach to curriculum delivery is the basis behind some of the new wave learning theories and I feel we have been doing it for years. Also, technical applications of Venn diagrams and a range of tables and graphs are being included in integrated programs as if they have suddenly been discovered. Both of these have the potential to improve student learning.

As teachers of mathematics however, I feel there is a need to identify which are the key mathematical processes before they are randomly chosen for these integrated programs and hence in this section of the workshop we will make a list of processes and discuss their worth.

At present, though most of us teach mathematical processes in the mathematics classroom, the processes involved with functions and relations (including the use of formula) form a large part of the courses. There are also mixtures of ideas such as probability functions and vector functions or the formulas of rectilinear motion and evaluation of rate of change by differentiation.

While mathematics teaching is not just about vectors, sets or functions, we do need to highlight the precise nature of the definitions of the mathematical ideas. We routinely develop mathematical concepts and thought processes in a stimulating but organised way. Our planning is intuitive.

However, do we tell the students the steps in our lesson structures, and if we are going to tell them, how do we do it?

Organisational models

Many people in the corporate world use these models. For example the organisational model of ‘Mission > Roles > Goals > Plan’ is one of the most common organisational models that is used in business plans (Covey, 1989). Even the Education Department of Western Australia uses a variation of this model in the headings of its Strategic Plan Table 1 is intended to identify links between different educational applications of models.

Table one

The organisational model as per the Covey reference	The EDWA Strategic Plan	A general lesson plan format	WA year 12 Modelling with Mathematics subject outcomes
Mission	Our purpose	Desired outcomes	Clarify a problem
Roles	Our values	Materials required	Choose a model
Goals	Key objectives	Content objectives	Use the model
Plans	Major strategies	Lesson format	Interpret and check

The eight point model in the concept analysis section of this paper was developed using this group of models with step 1 ‘Develop the concept’ related to the Mission (from the first column of table one), step 2 ‘Establish the notation’ is the definition of roles, step 3 ‘Define the operations’ as a goal and step 4 ‘Practical applications’ as the plan. While these models are helpful, they are only a starting point for our own creativity. For example, I have added an extra four steps to the ‘develop a concept’ model. That is, step 5 Solve problems, step 6 Investigate the history, step 7 Identify people’s uses and step 8 Design models.

After evaluating many of these organisational models I have categorised them into three types, which I refer to as *Thought trains*, *Cognitive planes*, and *Automobilisers*. (as in models of trains, planes and automobiles.)

Type one: Thought trains

These are cyclic models that go round and round like a model train set. Action research is a popular example with its repeating cycles of Plan > Act > Research > Reflect > Plan > Act > Research > Reflect, etc. (Tripp, 1997). When developing function theory, I keep a picture of a triangle on the board with tables, rules and graphs at each corner and a double arrow on each side. Then during lessons I indicate to the students what we are doing with the function at the time such as developing a rule from a graph of displayed data.

Type two: Cognitive planes

These are hierarchical models that develop in steps where the order indicates increasingly higher levels of thinking. The first four steps of the Develop the concept model as discussed require increasing levels of understanding and should be done in order. That is you need to develop the concepts before practicing applications. Also Bloom's *Taxonomy* (Bloom, 1956) is an example that was recommended in the Education Department of Western Australia Talented and Gifted Students policy file. This model has the following levels:

1. Knowledge
2. Comprehension
3. Application
4. Analysis
5. Synthesis
6. Evaluation

and can be used for test construction, curriculum development and as a guide to move students from a lower to a higher level of thinking.

Type three: Automobilisers

These are the dot point models where order may or may not be important. The models in table one based on the mission, roles, goals and plan steps do not necessarily increase in degree of difficulty. For example, in the Year 12 Modelling with Mathematics course (from column four of table one) the students do not move up a grade as they achieve an outcome (as in the Student Outcome Levels).

This is not the only way to categorise organisational models however it is intended to stimulate discussion. As part of an ongoing interest in this field I am very interested in compiling a list of alternative models.

Summary

The core business of a mathematics teacher is to develop an understanding of mathematical concepts, processes and applications with their students and this workshop is designed to stimulate discussion and sharing of ideas on this core business via a fictitious *Test for Mathematics Teachers*.

The real test for mathematics teachers however, is in how we manage the development of mathematical processes and which models we choose when developing the concepts.

Finally we need to decide whether identifying these organisational models during lessons, will increase student learning.

A test for a mathematics teacher

Note: the first question is linked to the last, the second to the second last, the third to the third last, and the fourth to the fifth. It may be of interest to note that the third question, 'What is the shape and size of the number five?' has an almost identical answer to the sixth question, 'Does a cone have to have a base or can it be open like an ice cream cone?'

1. Define the term mathematics.
2. What is the most important number?
3. What is the shape and size of the number five?
4. What is the shortest distance between two points?
5. What is an angle?
6. Does a cone have to have a base or can it be open like an ice cream cone?
7. What is the second most important number?
8. Define the term learning.

The answers to these questions might be open to conjecture and it would fill a chapter to discuss each in its entirety. To satisfy inquisitive readers my brief answers are as follows.

1. Mathematics is a study of concepts, models and processes.
2. The number one is most important to me, for with it you can build all the other numbers.
3. The number five (like all other numbers) has no set shape or size. It is a concept.
4. This depends on the geometric space (e.g. planes fly curved paths on the earth's surface).
5. An angle is more a process of rotation, than an actual shape.
6. A geometric shape is also a concept and therefore can take many physical forms.
7. The number zero has special significance in allowing place value in our number system.
8. This is described in the previous chapter in terms of processes and mediums.

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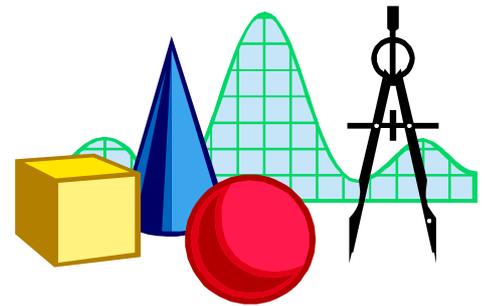
A Test for Mathematics Teachers

By Rory Lane

Section 1: Conceptual Analysis

Part A

1. Define the term mathematics.
2. What is the most important number?
3. What is the shape and size of the number five?
4. What is the shortest distance between two points?
5. What is an angle?
6. Does a cone have to have a base or can it be open like an ice cream cone?
7. What is the second most important number?
8. Define the term learning.



Hint: The first question is linked to the last, the second to the second last, the third to the third last, and the fourth to the fifth.

Part B

List as many mathematical concepts as you can eg vectors, sets, and matrices.

Part C

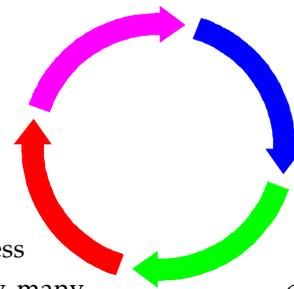
Evaluate the following procedure for the introduction of mathematical concepts.

1. Develop the Concept
2. Establish the Notation
3. Define the Operations
4. Practice Applications
5. Solve Problems
6. Investigate the History
7. Identify people's uses
8. Design Models

Section 2: Process Management

Part A

What is a function? For example, is it a set of ordered pairs, a process governed by a rule or a mapping from one place to another? How many functions are there? Can you have vector and matrix functions?



quadratic

Part B

List as many mathematics specific processes as you can eg differentiation, multiplication of two matrices and the drawing of a reflection.

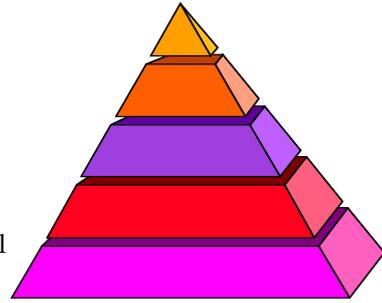
Part C

Do you use any models of processes when designing lessons or groups of lessons and do you identify them to the students?

Section 3 Organisational Models

Part A

What is the difference between cyclic processes, hierarchical taxonomies, and organisational models and do you identify these to students when teaching mathematics?



Part B

List any organisational models that you use eg Action Research, Blooms Taxonomy, and Problem Solving Models

Part C

There seems to be an abundance of new pedagogical models being put forward by popularised academics at the moment. However the idea of developing skills in students to work mathematically covers most of the elements involved in these models. Is the subject of mathematics getting the recognition that it deserves for the development of transferable skills in conceptual analysis, process management and the use of organisational models or is another layer of eduspeak just drowning us out?

Extended assessment tasks for mathematical inquiry in senior secondary mathematics

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Extended assessment tasks for mathematical inquiry such as investigations, and problems solving or modelling tasks, have been incorporated in high stakes senior secondary school based assessment around the world to foster high level mathematical thinking in ways that complement examinations. Important considerations for the development of such tasks include their nature, purpose and scope, mathematical content, contexts for application, and suitable assessment processes including authentication. In this paper we outline some recent developments around the world and discuss the recent transition in Victoria from the centrally set Investigative Project Common Assessment Task (CAT) 1 to teacher developed application tasks using task structure and context advice from the Victorian Curriculum and Assessment Authority.

The evolution of extended assessment tasks for mathematical inquiry in senior secondary mathematics

Rationale

The traditional model of assessment in senior secondary school mathematics in many countries has been, and continues to be a combination of tests and/or time limited examinations. The value of this form of assessment is its capacity to test a wide range of standard and non-standard routines, closely linked to the content of a prescribed course of study. A substantial component of teaching is typically directed at preparing students for these examinations.

As a result, this teaching is highly focussed with good teachers developing student's ability to handle a wide range of mathematical techniques and to apply these

techniques to in simple contexts and to some contexts where several distinct procedures have to be executed with varying degrees of mathematical sophistication.

One drawback of this form of assessment is that it provides a relatively weak platform for students to engage in extended mathematical problem solving or to undertake investigative work where the appropriate mathematical procedures are not necessarily clear from the start, but need to be selected, tried and adapted over a period of time. For this reason, assessment authorities in many countries have been exploring ways to expand the range and type of assessment tasks that taken together comprehensively address all the objectives of a senior mathematics course of study. This paper examines the evolution of extended assessment tasks using exemplars from several countries over the last ten years.

A fully prescribed task

Advantages

These tasks have high quality assurance for the mathematical content of the task and free teachers from having to worry about designing these tasks in the first instance. They also enable teachers to focus on the relevant mathematical processes and on implementing new ways of assessing student performance. They provide a strong impetus for teacher professional development and collaboration in implementing a new assessment regime.

Drawbacks

These tasks require a high level of involvement and scrutiny from the central curriculum and assessment authority. Teachers may be nervous about whether they have grasped the intended mathematical depth and breadth of the problem and whether they have identified all of the main solution pathways. Teachers may also be unsure about the extent to which they may reasonably assist students and worry about undue assistance in *other* schools, or from external sources such as coaching or tutors. These tasks are high intensity development projects for the central authority, with accompanying issues of security being paramount. Fully prescribed tasks expect that all teachers can uniformly and correctly interpret the mathematical demands of the task for students, and that teachers will not provide undue or unwitting assistance to their students. Tasks have a strictly limited life expectancy, although they may continue to live on as ‘rehearsal’ tasks or become adapted for similar work in earlier years.

VCE 1993 Change and Approximation: Common Assessment Task 1

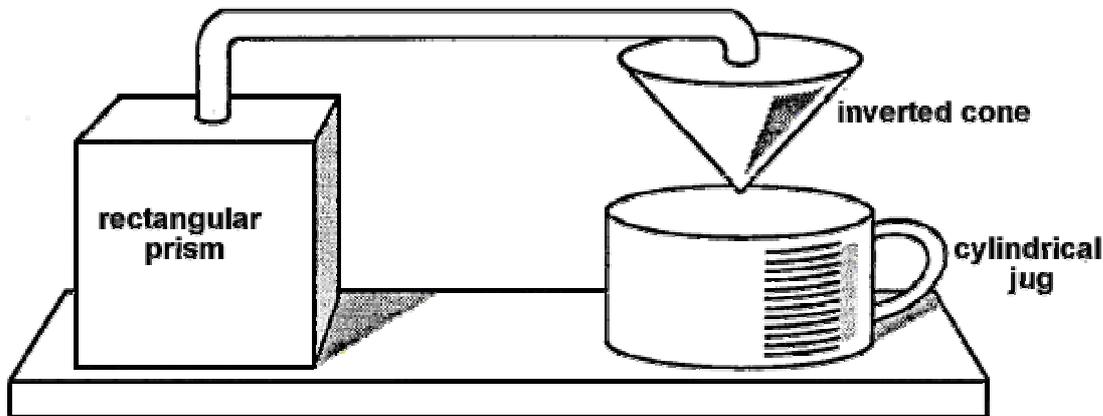
— Investigative project: Filter coffee-makers (Board of Studies)

In this starting point you should examine the connections between the flow of liquid in the different parts of a filter coffee-maker. You are to explore the relationships between the height and volume of liquid and how they vary with time.

A filter coffee system consists of three connected containers. Water is heated in a container, pumped through to the filter which contains the filter-paper and coffee-grounds and then drips into the coffee jug.

a. Working with a simple model

You should assume that water is initially drawn from a container which can be regarded as a rectangular prism and is pumped to a filter which has the shape of an inverted right circular cone. It then drips into a cylindrical coffee jug.



- i To begin, assume that water is pumped into the filter at a constant rate, and that the filter is blocked. That is, we are only looking at the first stage of transfer of liquid. Using a rate of flow which would allow a litre of water to pass into the filter in 15-20 minutes and by assuming that the filter is large enough to hold this amount of water, find the rate at which the volume and the height of water in the filter change with time. Sketch graphs of the rates of change which you have found.
- ii From these formulae for rates of change, you should now be able to find, by anti-differentiation or otherwise, formulae for volume and height in terms of time.
- iii Now look at the second stage of transfer of liquid. Imagine that the filter is full of liquid. Allow this liquid from the filter to flow into the jug without any more water being pumped into the filter. Assume that the rate of flow is proportional to \sqrt{h} , where h is the height of liquid in the filter at time t . Find rate of change formulae, with respect to time for height and volume of both the liquid remaining in the filter and the liquid filling the jug.
- iv From the various formulae for rates of change, you should now be able to find, by anti-differentiation, formulae for the volume and height of liquid in the jug in terms of time. Alternatively, sketch graphs of the rate of change functions to describe the behaviour of the system. (Board of Studies, 1993).

b. Extending the model

Here are a number of suggestions for extending the simple model given above. You are required to pursue only one of these or a similar alternative

- i Consider the whole system operating simultaneously. In particular this means looking at the height of liquid in the filter when water is pouring in and dripping out of the filter.

- ii You may wish to consider jugs of different shapes and/or the problem of where to place measurements showing cup gradations on the side of the jug. Which cup is filled the quickest?
- iii The assumption that water flows into the filter at a constant rate is questionable. In practice it starts slowly, builds up to a maximum and then decreases. Find a simple function with these properties over the domain under consideration and explore the effect of using this rate of flow on the rate of change of height and volume of water in the filter. (Board of Studies, 1993).

Comments

This investigation required students to examine the way in which different variables are connected in a mathematical model of a real situation. In particular, students were asked to consider the connections between the rates of change of different variables in the model. Some use of experimental data could help students to set realistic parameters, but experiment alone would not be sufficient.

In particular, teachers and students need to attend to the following important steps in developing a mathematical model:

1. Making assumptions that simplify a real situation. In the example above, they might ignore evaporation or assume a container to be a perfect cylinder; and consider the likely effects of these assumptions on results.
2. Identifying the key variables in a model. If there are too many variables to deal with initially, they might assume some of them to be constant so that they are left with a problem which they can work on mathematically. The choice of constants will, in general, be left up to students who need to give consideration to making realistic choices.
3. Having fully explored a model with these assumptions it may be appropriate to change one, or more, of these assumptions.
4. At each stage, it is essential that results are evaluated and interpreted. In particular students must consider carefully the domain in which their results might be valid and consider what might happen with extreme values of the variables.

Model tasks provided

This approach is sometimes a first and necessary step, where, as in the International Baccalaureate, students in consultation with their teacher have to choose a topic for investigation (see IBO, 2000). The use of model tasks can also be typical of a second stage after teachers have had experience in using fully prescribed tasks. Model tasks make sense only when interpreted in the light of well designed and explicit criteria for assessment, which are a strong feature of the IB.

Advantages

These tasks help teachers to strengthen their ability to prepare their own tasks, possibly by adapting or modifying or extending model tasks; provide scope for local variation; and so reduce opportunities for undue assistance from outside sources. They provide students with ideas for possible investigations and also indicate the breadth and complexity expected of them. Model tasks continue to be viable during the lifetime of an accreditation period and, in contrast to fully prescribed tasks, there are no associated security issues.

Drawbacks

These tasks assume that key mathematical features implicit in the model tasks are obvious to teachers, and assume that teachers are able to design tasks of comparable richness and rigor. These tasks assume that teachers are proficient in applying assessment criteria. They place a high demand on teacher's competence in choosing topic areas and developing a sound framework for investigation.

Absorbing Shocks (IBO Mathematics HL, Portfolio 2000)

Springs and shock absorbers in cars are designed to provide comfort to passengers, and stability on the road for the car. Engineers have designed a new type of shock absorber, and we have the task of testing it.

The following test is devised. The front of the car is pushed down 10cm from its rest position and then released. The displacement of any point on the front of the car from its rest position is measured as a function of time. The car rocks up and down and then returns to rest.

1. Sketch what you think is the graph of the motion.

The maximum distance that any point on the front of the car travels **below** the rest position **after** being pushed downwards is used as a measure of the effectiveness of the shock absorber. Dividing this distance by the original displacement, in this case 10cm, gives the rebound ratio. A ratio below 1% is considered acceptable. For this particular type of shock absorber, the distance y from the rest position is modelled by the function $y = -10e^{-10t}[\cos(10t) + \sin(10t)]$ where t is measured in seconds and y in centimetres.

2. Sketch a graph of this function.
3. Comment on the suitability of this function as a model for the motion.
4. How long does it take the front of the car to reach the rest position for the first time?
5. Find the velocity at that point.
6. How long does it take the front of the car to travel below the rest position for the first time after being pushed?
7. How far does it travel below the rest position?
8. Is this shock absorber acceptable?
9. How long does it take the front of the car to reach the rest position for the second time?

10. Find the velocity at this point.
11. Estimate the time it takes for the front of the car to reach its lowest point after it reaches the rest position for the second time. How far below the rest position is this?
12. Comment on your results.

(International Baccalaureate Organisation , 2000).

Comments

The important features of the graph are not readily identified analytically, so a graphical approach supported by the use of technology, including careful consideration of domain and range, is required. A sound understanding of the general behaviour of the function is needed for students to be able to identify key features of the graph and carry out related analysis .

A clear understanding of the construct of ‘rebound ratio’ is required as well as the ability to interpret this in context. Students need to be able to identify the location of the relevant local minimum (determined numerically), which is not obvious. They then need to use the corresponding time value, $t = 0.628319$ (6 significant figures), to evaluate the rebound ratio $\frac{y(0.628319)}{10}$ and conclude that its magnitude is less than 1 % for the shock absorber to be acceptable.

Prototype tasks with matching assessment criteria

A prototype task includes explicit guidance on constructing an extended assessment task so that, when adapted by users to their local circumstances, it will minimise the ‘design load’ on teachers so that they can benefit as much as possible from others’ successful experience (cf. Burkhardt, 2002).

Advantages

Design brief draws the attention of teachers to key mathematical features needing to be embedded in tasks, such as the Victorian Certificate of Education (VCE) application task (see VCAA, 2001) but also recognises that clear pathways and choices of pathways need to be presented to students in order to foster a depth and breadth of mathematical investigation. Prototype tasks allow for wide local variation in contexts and parameters and seeks comparability of task demands through the use of outcomes with associated ‘key knowledge and key skills’, and through the use of assessment criteria. These tasks do not have associated security risks that fully prescribed tasks have, and are useful over an extended period of time.

Drawbacks

These tasks are not an easy first step for teachers to implement. The expected standard is generally described in terms of the outcomes, rather than through detailed examples of sample tasks, and hence places a greater responsibility on teachers for understanding of, and fidelity to, expected standards. Equally, there is an obligation the

central authority to moderate or audit standards and provide feedback to teachers on acceptable implementation, as well as *advice* on task design.

Further Mathematics — a data analysis application task

Component 1: Displaying and organising univariate and bivariate data.

Component 2 : Consideration of general features of the data.

Component 3: Undertaking analysis of the data such as regression analysis, the use of transformations to linearity, de-seasonalisation or analysis of time series.

Starting Point 2: Investigate road fatalities by road user type

Source: Australian Transport Safety Bureau. The Victorian Curriculum and Assessment Authority gratefully acknowledges the permission of the Australian Transport Safety Bureau for the use of this data.

Some possible aspects of data analysis that an application task based on data set 2 might incorporate are:

Seasonality

Consideration of whether the number of pedestrians (or car or motorcyclists or bicyclists) fatalities has, or seems to have, a seasonal component.

- selection of three years at random (these need not be consecutive years);
- graphing the data for each of the three years (fatalities against month) on the same time series plot (that is, trend lines on top of each other);
- comment on seasonality or otherwise;
- repetition of analysis with the fatalities expressed quarterly rather than monthly, comment and comparison with monthly figures;
- determination of seasonal indices;
- deseasonalisation of data.

Trend

Teachers may wish to deseasonalise data, as appropriate, before students undertake their data analysis.

To examine trend any of the following could be included.

- selection of three years consecutive monthly data for any of the variables, and construction of a time series plot, with associated regression analysis;
- consideration of annual totals for any of the variables over the entire period, and construction of a time series plot, with associated regression analysis;
- selection of one month, for example January, and consideration of this month for every year of the entire period, construction of a time series plot, with associated regression analysis.

Ratios

- consideration of the number of motorcycle fatalities as a proportion of the population, or of the number of motorcycle registrations (this data can be found from the data sets supplied on the VCAA website from 2000);

- consideration of the trend in this ratio over time;
- consideration of which provides the better explanation of how things change, and explanation of this analysis.

Correlation

Where time series are involved, it is important to use annual or deseasonalised data, unless the seasonal patterns are the same for the two variables selected for analysis.

- construction of a scatterplot for the number of fatalities (for example, for the categories of pedestrian and bicyclist);
- description of the relationship between the variables involved;
- calculation of the correlation coefficient, the coefficient of determination, and the regression equation. Interpretation of each of these for the given context.

Other sources of data

Teachers are encouraged to obtain data sets from a variety of sources, and to develop their own application tasks based around either the suggested theme and starting points, or a theme of interest to themselves and their students using their own starting points.

(Victorian Curriculum and Assessment Authority, 2001).

Comments

This prototype task differs both a fully prescribed task and a model task in that it does not provide explicit detail about the specific elements of the task, but rather indicates to teachers key elements which should constitute the extended investigation in the given *application context*. Each year similar advice is provided by the VCAA outlining a suggested theme and several possible starting points with their own context. Within this framework teachers are expected to embed into each of the three components detail related to the application context they have chosen, or develop like tasks using their own theme and starting points.

Some key design principles

Whatever model, or variation, is adopted some key design principles need to characterise good extended assessment tasks for mathematical inquiry in senior secondary school mathematics:

- tasks should have a clear and substantial link to specific mathematical content prescribed for courses and require students to directly utilise related knowledge and skills;
- tasks should provide opportunities for graduated levels of response in mathematical application, analysis and interpretation;
- tasks should be broadly accessible to the student cohort with scope for clearly differentiated levels of performance in terms of the assessment criteria;

- tasks should be seen as directly rewarding for students in providing genuine applications of mathematical knowledge and the opportunity to make connections both within mathematics and its application to the real world;
- general assessment criteria should be provided for assessing student performance and, where possible, task specific advice should be provided;
- the mathematical integrity of the task should be transferable across local adaptations and variations;
- consistency and comparability of assessments should be supported by the use of assessment criteria, in conjunction with advice and monitoring by the central authority;
- public credibility of assessments should be safeguarded through mandated requirements for authentication of student work through, for example, the use of log books, or conferencing with teachers, or oral presentations, or confirmatory tests based on content central to and learned from the investigation, or through statistical and other forms of moderation.

Conclusions

While the move from only using timed tests or examinations to a broader assessment repertoire has not been easy and, at times, has been controversial, there is now a well established and substantial body of international experience on which future developments in this area can be built. Within Australia, these experiences have been demonstrably successful in several states and territories. Notwithstanding different starting points across Europe, extended assessment tasks, for example, in England, Denmark, and the Netherlands, now count for a significant element of a student's final result. These trends have been confirmed by the relatively recent adoption by the International Baccalaureate Organisation of extended assessment tasks within its portfolio work for all IB mathematics courses.

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Strategies for engaging students in mixed-ability classrooms

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General overview of the workshop

The idea for this workshop has come from a long held interest in engaging every student in the class and understanding how I as a teacher cater better for some individual and groups of students than others. The work this year was in three main projects:

- A *Senior Years Project* — observations indicate that many teachers at this level use a limited repertoire of actual classroom teaching and assessment strategies.
- A *Using Technology in Mathematics Project* — where a concern is that technology is used in all learning areas, but that quite often excellent practice and methodologies in other learning areas do not transfer to the mathematics classroom.
- As a mentor, observer and assessor for teachers engaged in a Graduate Certificate of Numeracy — where it becomes obvious that good classroom practitioners generally have a range of strategies that they consistently use and adapt to engage the students in whatever they are doing. Students are encouraged to have inquiring and open minds and at the same time the teachers are clear on what it is they want to achieve in terms of learning outcomes.

There is also evidence that most teachers have about 6 different strategies, but when the pressure is on, will revert to one or two familiar strategies.

From these experiences and observations I have put together a range of strategies, some of which are new and others which have been adapted from the work of others. With all strategies I have tried to include a range of ways of working, so that the methods are flexible and adaptable to the learning styles of a wide variety of classes.

Number revision

How many ways can you get this answer?

Ways to engage all students in this problem could include:

- Today you have to do ... problems;
- Make sure you have at least two: addition problems, multiplication problems, division problems or a problem that uses three or more numbers and at least two signs.
- Do a really hard and complicated problem, write a problem that uses words to get this answer or can you do an area or perimeter problem that gets this answer?
- Have students take it in turns to choose the number and the variations.

Guess my number

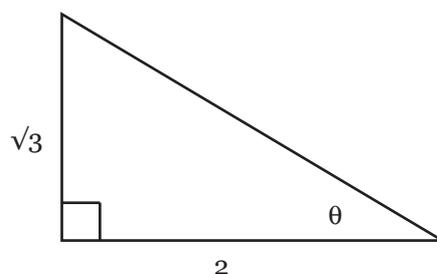
- Vary the range that the number is chosen from.
- Ask students to use particular language such as odd, even, larger than, smaller than, multiple of, square number or perfect square, prime number, etc.
- Have students take it in turns to choose the number (or work in groups), then as the other students guess the students record the guesses using mathematical signs; e.g. $<$, $>$, $=$, and not. Over time introduce variables.

Open and closed problems

Often in mathematics questions are asked so that there is only one answer. When students are beginning to understand a process it is sometimes beneficial to frame the question in an open way to allow experimentation and trial and error approaches to encourage engagement with the process.

- Using brackets how many answers can you get to the problem $3 + 4 \times 5$?
- Can you make all of the digits using only the numbers in 2002 and any signs, brackets or indices?

Find θ **OR** write down everything you can about this triangle.



Brainstorming

Documentation of a process of brainstorming and two adaptations that are useful in the classroom.

Ask each person what they expect to get from the workshop. Write this on a whiteboard, select new things and organise as appropriate. Proceed to introduce your topic and do the presentation. At the end of the session return to the list on the whiteboard and get agreement from the group to remove those things that have been well covered, identify those things still to take action on and leave those things that still need to be addressed.

As a part of the process 'stickies' can be left on each table and people invited to record questions, issues and feedback. These can then be put in a prominent place and addressed again at the end of the session.

It is possible that this method can be used in this form or adapted for use in classrooms.

Classroom applications

(These do not have a real world context, but are used as illustrative examples of the use of the methodology)

The class brainstorm

I will explain this using an example. When teaching year 9 or 10 coordinate geometry I stand at the white (or black) board and invite students to let me know everything they know about coordinate geometry. Sometimes a little prompting is needed, such as 'you know x and y coordinates', or perhaps draw some axes on the board to start the conversation. Then just wait patiently for suggestions to come. Write up all of the suggestions that fit what you want to do and explain fully any that you do not write up. On nearly all occasions I have done this I have got everything I needed except perhaps the equation of a straight line. Any thing that does not come up I write on the board and put a circle around. I then say something like: Over the next ten lessons (or whatever) you need to teach each other all of this stuff and find ways to check out that everyone has learnt it. I will then teach the stuff that is in the circles. On most occasions this is where I encourage the students to find 'real' contexts for what they are doing. The first time I did this I was amazed at what I had written on the board and the enthusiasm with which the students organised themselves to do what I had asked. I was pleasantly surprised also by the results in my very traditional test on that first occasion.

Write down everything you know about...

This is one I use on many occasions to find out more about students prior learning. In particular the first week with my Stage 2 Mathematics 2 class was always spent 'writing down everything you know about quadratics'. This can be done using the think, pair, share strategy but usually I let them sit silently for quite a while thinking and then use the blackboard to get each student's ideas, knowledge down. Then very much directed by my questions I would make the links to a whole lot of algebraic manipulation

(including the difference of two squares $x - y$, factorising $a \sin^2x + b \sin x + c$, etc.), graphs (connections between $y = ax$, $y = ax^2$, $y = a2^x$, etc.), roots, solutions of equations algebraically and graphically and etc, etc, you know it all better than me. Doing this I found was a confidence booster and allowed me to make connections to a lot of stuff we would be doing and the algebraic manipulation of that stuff.

An application of the Think, Pair, Share methodology or the mathematics equivalent Think, Double, Square

An application this can be used for is the setting of class tests.

Think

Firstly ask every student to set one question on the work or topic that you have been doing for the past week, month, whatever is appropriate.

Give guidance for the setting of the question, such as make it a question worth about five marks on what you consider to be the most important thing you have learnt in the past week. You must be able to present a solution with marking scheme.

This can of course be varied, for example:

1. Set your question on the thing you found easiest/hardest/most interesting/most challenging over the last week.
2. Set your question so that a connection between a graph/diagram and algebra/evaluation is involved.
3. Make it a multiple choice question or a short answer question or a two step problem or one problem that incorporates everything for the topic.

Double

Then ask the students to work as a PAIR and make a decision on the best question for a test OR a way of writing a new question that they think is better than both original questions. Solutions to original questions must be checked together and any new questions must have solution and mark scheme.

Square

Each PAIR then PAIRS up with another PAIR to make four students in each group and they work through the process again. That is they check answers from each pair together, and decide on one question to be a part of the class test with solution and mark scheme.

Presentation

Ways to vary the presentation of the questions.

Sometimes put them into a class test.

On other occasions collect the questions, make four copies and the next lesson reorganise the groups so that students explain their question to a new group of three students.

Sometimes get each group to present to the class and have a class discussion about the question and then either revisit the topic or set the test.

Sometimes ask the students to reflect on what they had learnt from this process either as part of a class discussion or for some structured writing.

Discussion

What are other ways the Think, Double, Square method can be used?

Content, context and competence

These approaches are based loosely around a point that William G Spady (a Learning to Learn consultant) makes that ‘you can know the content, memorise the facts and practice in as many situations as you like, but you do not know you are really competent until the context arises that you need to demonstrate your content knowledge and competence in’. My argument is that if you develop the concept understanding as well as the content skills, competence in any context is more likely to follow.

Teaching a computer to count

With lots of topics in mathematics and in many mathematics lessons the first thing that happens is to get the formula and then use it to solve problems. Just substitute in the numbers and evaluate, get the answer and move onto the next problem. This can lead to confusion over which formula to use and when, or worse still not being able to remember the formula. What would happen if you had to develop a topic without using the formula right from the beginning? The idea here is to develop an algorithm and then use it to develop students understanding. The final aim is to make a stronger connection between the underlying concept (using the algorithmic approach), a visual representation and the formula.

Based on a paper that John Rice of Flinders University wrote on teaching a computer to count, you can develop any range of concepts. I have worked with two colleagues to do this for addition and multiplication patterns (arithmetic and geometric progressions). An approach is to get students to describe patterns in pairs. The students sit back to back, one with some patterns and the other with a blank piece of paper. Then the student with the pattern describes to the other student what it is they have to draw or write. First time you do it there are no language limitations, then you start to specify constraints such as: you must describe the pattern in three sentences; you can only use the following language to describe the pattern. Then use spreadsheets and algorithms to describe the pattern. Algorithms only do a number of things. They must have a starting point and a count, can have gates and can have loops. It is not necessary to develop fully the algorithm, rather to give the idea of using limited language to describe a process. The aim of doing this is that the students get to the concept along with developing a range of methods of describing the pattern.

Using a range of introductory contextual problems

A common approach to teaching mathematics is to develop the skills first and then to solve problems after the skills development has occurred. This approach provides a list of problems first. Quite often the problems can be solved by mathematics already learnt.

A possible approach is to write 20 problems on the topic, ask the students to go through the problems and choose five that they will attempt to solve, by any means, and also to classify the other problems in some way. Instructions re classifying can be very open or quite explicit, for example: classify the problems as addition or multiplication problems; classify the problems according to whether you have seen this type of problem before or not; classify the problems by writing the first step you will use to solve the problem.

This approach has a number of benefits. It connects with student's prior understanding, it provides a context for the development of the mathematics, it allows for discussion of the reasons for developing the mathematics and hopefully encourages greater engagement of the students in the topic.

Note: This approach can be used in a game of 'Beat the Textbook' which has the purpose of students making the connections between all the different parts of the chapter. Can be use to predict what part B is about or to discuss the connections, similarities and differences between parts A and B.

An example is the midpoint

Coordinate geometry at year 9 and 10 seems to be an area particularly susceptible to confusion of formula and difficulties with what are particularly simple concepts, but are now being encountered in a new context. Perhaps it is because these are simple concepts that we consider we can get straight into the formulae. A better approach may be to ensure the concept and the language are fully developed. Using the midpoint as an example, I think students will have come across the idea of middle, centre, halfway and average in many different situations. Many times the connections between these ideas and the actual midpoint are not made by students. A possible approach is to get the students to develop the concept in a whole range of ways first and allow them to link their current understandings of middle, centre, halfway and average to the new context. This can be done in any number of ways.

Activity

1. Design a number of ways to do this.
2. Describe the positives, negatives and interesting points to this approach.

Why are we doing this project?

(And related questions such as: Where will I ever use this? Where will I use this in everyday life?)

Process

1. *History*: Find out who developed this; what they did; what were they working in response to. Find out about three other important things that were happening in world history at this time.
2. *Practical uses*: Why are we learning it now? (Connections to subsequent years, topics, other learning areas and media.) What is it used for now? (Other learning areas, practical uses.) Who uses it now? (Trades, mathematics teachers, scientists, accountants, etc.) Is this problem done more easily now using some form of technology? (Spreadsheet, graphics calculator, etc.)
3. *Worked example*, writing to explain every step. (An example is provided for the students.)
4. *Make up your own problem*, do as for 3. Get someone else to solve your problem. Get feedback on your problem (ease, topic, clarity). This can be creative, practical, usually creates a struggle to find numbers that work out and can create much discussion.
5. *Presentation*: Negotiate or vary the presentation as a project, assignment, essay, oral, poster or website.

Assessment ideas

This is a list of assessment ideas, some of which come out of the methodology suggestions. The list is intended as possible stimulus to encourage thinking about ways that students can provide their own evidence of their learning. I do not intend to discuss it, but will give explanations if anyone is interested.

- Class tests
- Brainstorming, before and after record keeping
- Student surveys
- Student record keeping
- Develop Product Analysis, Conferencing, Observation
- Open ended questions — Core and Extension
- Close activities
- Multiple choice questions
- Student writing
- Develop your own problem
- Can you teach someone else how to do this?
- Why are we doing this? — Project
- Beat the textbook or what happens next and how does it connect to what has been done (looking for similarities and differences) — predictive questions
- Prior knowledge — Where have you seen this before? What is it currently used for? What do I still need to learn?
- Doing the problems first. Encouraging students to find their own similar problems. Students explain why their problems are similar.

Approaching computation through mental computation*

Alistair McIntosh

University of Tasmania

The paper describes the background to current developments in the teaching of mental computation and informal written computation in classrooms, and describes how children's mental computation can be linked to their development of informal written computation, drawing on the experience of a project involving nine schools in Tasmania.

Background

Since the early 1980s, coinciding perhaps with the publication of the *Cockcroft Report* (DES, 1982), increasing attention has been paid to mental computation in many countries, particularly in the early years of schooling. Reasons advanced for this increasing emphasis include: that most calculations are done by adults 'in the head' (Wandt & Brown, 1957; Northcote & McIntosh, 1999); that mental computation develops sound number sense (McIntosh, 1990); and that mental computation promotes success in later work (DES, 1982). In Australia this emphasis has been endorsed by the *National Statement on School Mathematics* (AEC, 1981) and the National Profiles (Curriculum Corporation, 1994), and in Tasmania by the KINOs (DECCD, 1997) and the K-8 Guidelines (DEA Tasmania, 1992). Large-scale system based projects including Count Me In Too (NSW), Early Numeracy Project (Vic.) and First Steps (WA) all appear to endorse this emphasis.

It is probably fair to say that there has been little or no opposition to a greater emphasis on mental computation in theory, and that while perhaps still a majority of primary schools have not yet been touched by this movement, an increasing number of schools are beginning to develop teaching approaches based on discussing, encouraging and teaching flexible mental strategies, and some have moved a long way down this path.

This mirrors similar movements, with greater or lesser central support, in other countries including the United Kingdom, the United States and the Netherlands.

* This paper has been subject to peer review.

This change of emphasis is not difficult to justify or to translate into practice in the early years. However it faces teachers with a real dilemma in the middle primary years, when traditional practices involving the teaching of standard formal written computation algorithms become prominent. Suppose for example that a child, calculating $27 + 36$, has been accustomed to think either of mentally adding the tens ($20 + 30 = 50$), adding the ones ($7 + 6 = 13$) and then adding 50 and 13 to give 63; or alternatively of starting with 36 as the larger number, mentally adding 20 to give 56, and finally adding 7 ($56 + 4 = 60$, $+ 3 = 63$). Both these strategies are efficient and effective mental strategies, commonly developed, understood and used by children and based on 'natural' understanding of the way addition and place value work. However neither strategy is close to the current formal written algorithm, with its emphasis on placing the numbers in columns, always starting with the units digit, and 'carrying', and, above all, with the implication or even the overt teaching that 'there is only one correct method'.

What is the teacher to do when children are faced with additions where the digits are difficult to hold in the head while calculating? Should the teacher wipe the slate clean and, ignoring practices and understandings related to mental computation of smaller numbers which the child has acquired, teach the 'standard formal written algorithm' which approaches computation from a different and less 'natural' perspective (Plunkett, 1979)? Or should the teacher build on the practices and understandings related to mental computation of smaller numbers that the child has acquired, and help the child to develop extensions of these practices involving informal jottings and personally developed procedures?

International advice and practice is still evolving, with research on the development of non-standard but formal algorithms being developed particularly in the Netherlands. Some experts in the United States provide evidence that the teaching of standard formal written algorithms is counter-productive (Kamii & Dominick, 1997). However the Australian and Tasmanian documents all take the same basic stance:

Young children should be encouraged to explore and invent informal paper-and-pencil methods to supplement mental and calculator methods (*National Statement*, p. 111).

They [Band B students] should devise a variety of paper-and-pencil computational strategies, record the results of their thinking in their own way, and progress towards efficient, though not necessarily standard, procedures (*National Statement*, p. 117).

At Level 2, a student: 2.16 Uses a variety of strategies, including regrouping, to assist in adding and subtracting whole numbers when unable to compute mentally (*National Profiles*, p. 42).

At Level 3, a student: 3.16 Uses understood written methods to add and subtract (*National Profiles*, p. 58).

People use pen-and-paper strategies to extend the range of calculations beyond those easily performed mentally... However, people with good number skills use a variety of pen-and-paper strategies, which may not always be standard algorithms taught in classrooms. Rather, they use a variety of 'back-of-envelope' methods

which are appropriate (*Tasmanian K–8 Guidelines, Number Strand Overview*, p. 9).

Some pen-and-paper computational techniques are useful in daily life and, therefore, efficient methods should be learned. There is evidence to suggest that there are advantages in delaying the teaching of these techniques... (*Tasmanian K–8 Guidelines, Number Strand Overview*, p. 10).

Teachers should... avoid imposing their own preferences (the ‘standard’ algorithms) too early. Children can and do offer their own methods of calculation... It is the teacher’s role to encourage the students to refine their methods or develop different, more efficient methods (*Tasmanian K–8 Guidelines, Number Strand Activity Booklet 2*, p. 9).

In Mathematics classrooms, written computation techniques are receiving increasing attention. Some students may be progressing toward formal representations. However in situations where students choose to use written techniques, the written strategy may be unlike the formal forms of the mathematics classroom (*Tasmanian Key Intended Numeracy Outcomes*, p. 66).

This representative selection of quotations from Australian and Tasmanian system-level documents all deal with the interface between mental and written computation and all advocate or describe the development of informal, non-standard, written techniques for computation, springing from mental computation techniques.

This movement is not confined to Australia.

There is little doubt that one of the main reasons for our [UK] underperformance in the number tests of international surveys is the very early introduction of formal written calculation methods in this country... In 1998 the idea of ‘partial written methods’ was introduced into the National Numeracy Project Framework at Key Stage 1. This coupled with modifications to the National Curriculum Mental Tests made that same year, legitimised the use of ‘jottings’ as an aide-memoire in mental calculation (Thompson, 1999).

Although these statements, recommendations or prescriptions are being made not only in Tasmania but throughout Australia and internationally, and although they concern what has traditionally been the heart or core of numeracy/mathematics in the primary school, remarkably little evidence or documentation is available either to exemplify the processes of developing these ‘informal written procedures’ or to indicate the effects. There is clearly widespread agreement that this is the direction in which schools should move, but there is very little available which documents, illustrates or evaluates the journey. Indeed there appears to have been no project in Australia that has explored this critical issue.

During 2001–2 the Department of Education (Tasmania), The Catholic Education Office (Tasmania) and AIST (Tasmania) in conjunction with the University of Tasmania, has been involved in *Developing Computation*, a Commonwealth funded Strategic Numeracy Research and Development Project, to explore these issues in relation to the operations of addition and subtraction.

Developing computation project

The project, involving thirty-three Grade 2, 3 and 4 classes in nine primary schools across the three systems, has been conducted in four stages, which are shown in Table 1.

Table 1. Four Stages of the Developing Computation Project.

Stage 1 (to end of 2001)	Teachers work at developing mental computation strategies, to familiarise themselves with the strategies and to build up confidence and competence in their students.
Stage 2 (Jan – April 2002)	Developing addition and subtraction of one- and two-digit numbers. No teaching of formal written algorithms.
Stage 3 (May – Sept 2002)	Developing addition and subtraction of one- and two-digit numbers. Developing informal written methods involving numbers of two or more digits. No teaching of formal written algorithms.
Stage 4 (Oct – Dec 2002)	Developing addition and subtraction of one- and two-digit numbers. Developing informal written methods involving numbers of two or more digits. Individual school decisions regarding teaching of formal written algorithms.

During the first year of the project, attention was focussed entirely on strengthening children's mental computation, concentrating first on single-digit addition and subtraction, and moving, if and when children were ready, to mental addition and subtraction of two-digit numbers.

Teachers were given a package of carefully sequenced material aimed at moving children on from a reliance on counting in ones to the adoption of other more sophisticated and efficient strategies such as doubling, using near doubles and bridging ten. Great emphasis was placed on encouraging children to articulate their thinking and appreciating that calculations can be performed in a variety of ways.

Mental addition and subtraction of two digit numbers were developed through a careful analysis of the subskills needed, and developing teachers' familiarity with the range of mental strategies likely to be used by children, including starting from the left, adding on (for subtraction) and compensating. Throughout, children were encouraged to describe how they calculated mentally, and to listen to, and critique, other children's methods.

Figure 1 shows the project structure for the development of addition and subtraction.

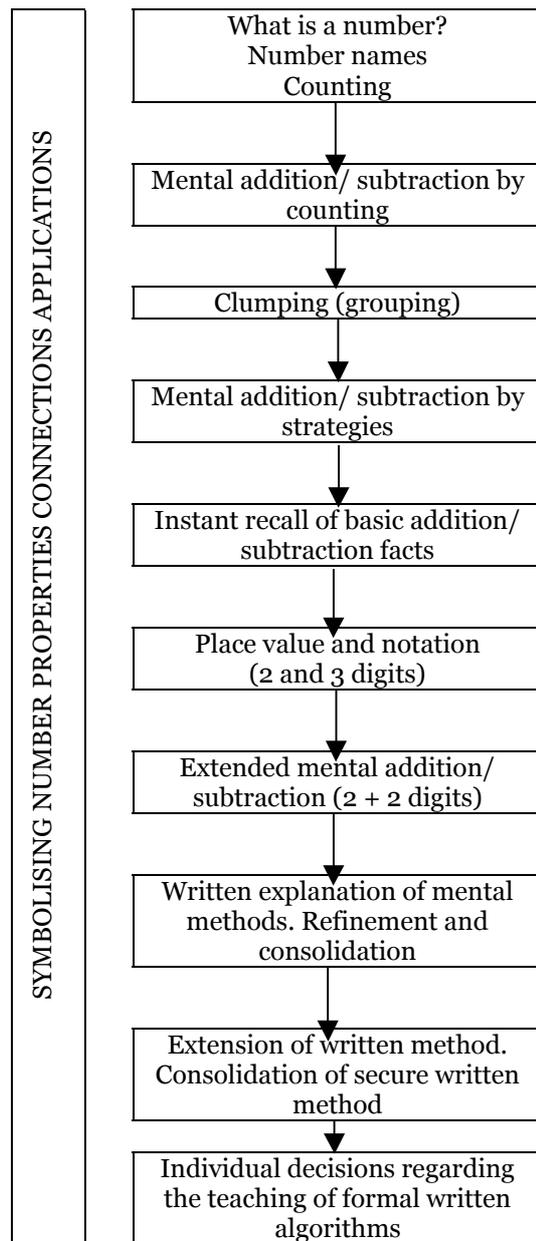


Figure 1. The development of whole number mental and written computation (addition and subtraction)

From mental to informal written computation

In the second year of the project all teachers in the project had agreed not to teach any formal written algorithms for the first two terms (Tasmania has a three term year) and to develop children's informal written methods, using as a springboard a developmental sequence proposed by the author. Table 2 shows the six-stage developmental sequence used by the teachers as a basis for developing informal written computation

Table 2. Development of informal written computation from mental computation.

Stage	Process
1	Strengthen children's mental computation with two-digit numbers
2	Encourage children to explain their mental method using paper and pencil
3	If method is 'sound', conference and refine recorded explanation.
4	Strengthen this method on examples of a similar difficulty
5	Extend its use to more difficult calculations
6	Consolidate it as an 'understood, secure written method'.

To determine whether a method was considered 'sound', that is, worth developing, criteria adopted from Campbell, Rowan and Suarez (1998) were used. The method had to meet three criteria: it had to be *efficient*, it had to be *valid* (that is, mathematically correct), and it had to be *generalisable*.

Figures 2, 3 and 4 show pieces of work from one child corresponding to stages 2, 3 and 5 in Table 2. Figure 2 shows a child's written explanation of their method, Figure 3 shows a second explanation of a similar calculation after refinement, while Figure 4 shows the same method being used for a more difficult calculation.

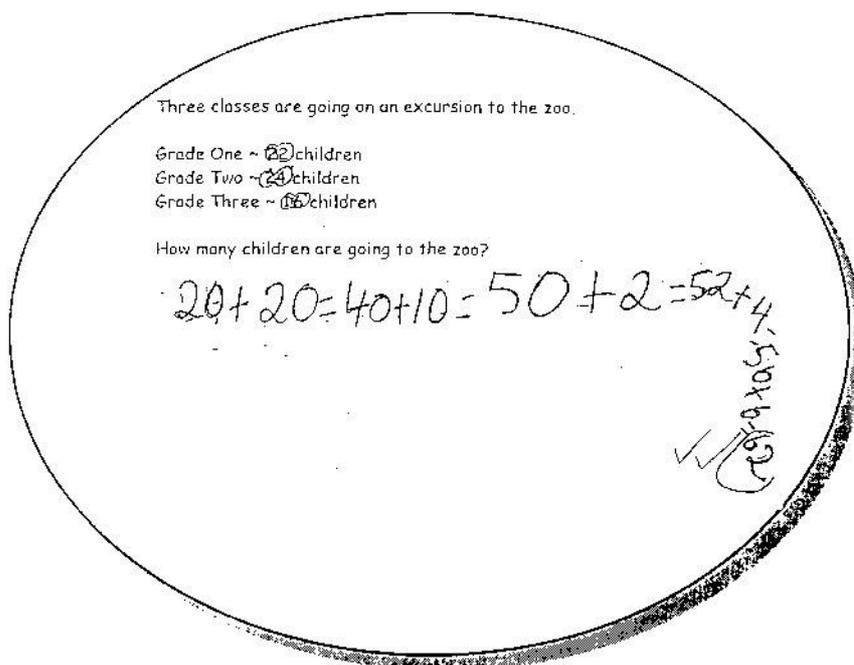


Figure 2. A child's written explanation of his mental calculation method.

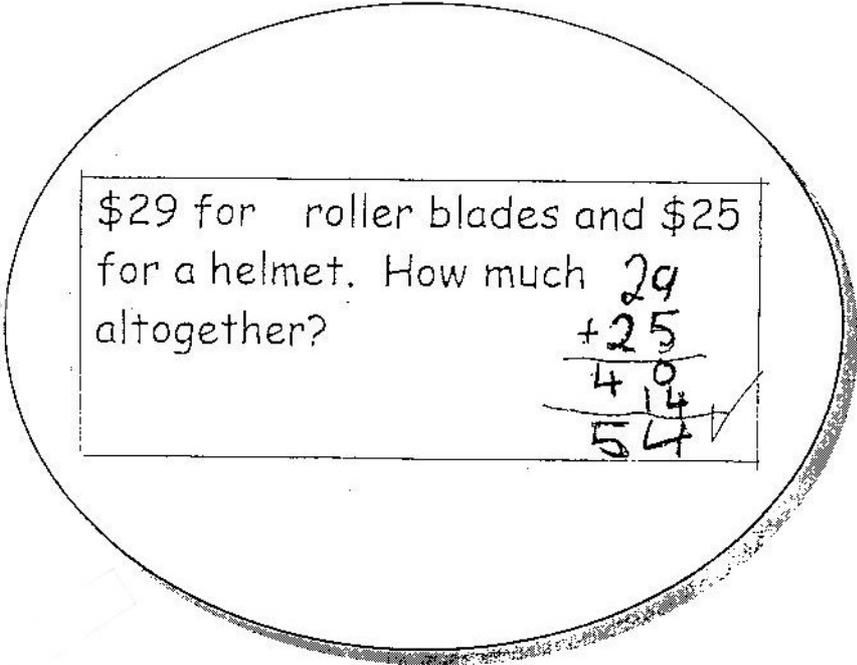


Figure 3. The same child recording a similar calculation after 'conferencing'.

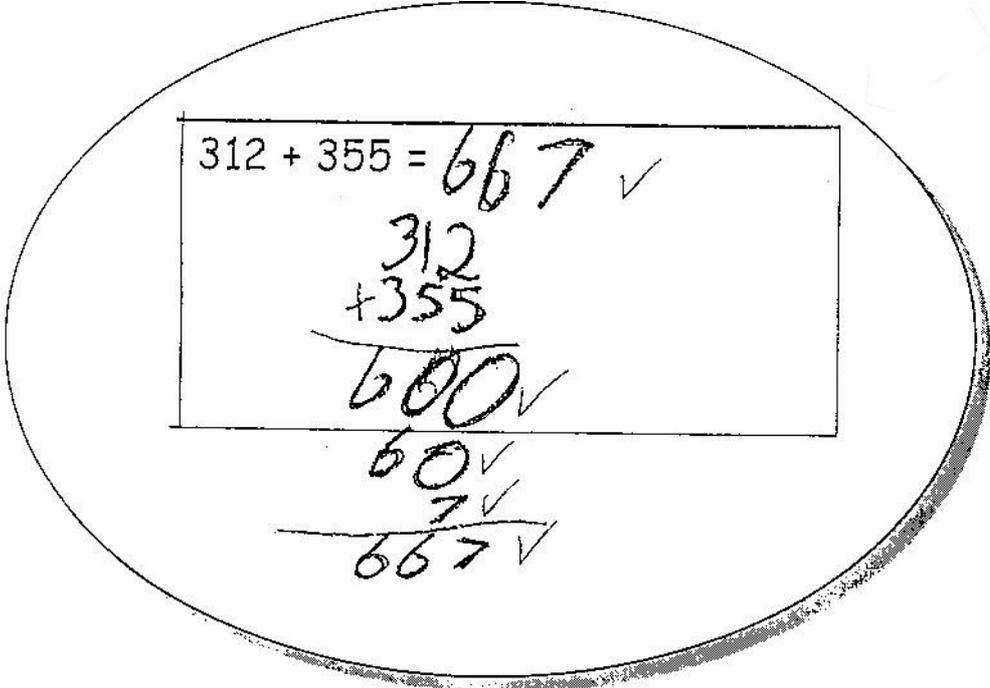


Figure 4. The same child using a similar method to compute a more difficult calculation.

The method described above was not adhered to strictly, rather it formed a very valuable framework in developing a connected transition from mental computation to informal written computation.

While the project is not yet complete, significant findings thus far are:

- all teachers in the project agree that the concentration on mental computation has greatly increased children's competence and confidence in handling numbers and in understanding place value;
- all teachers in the project would now recommend delaying the teaching of any formal written algorithms — in most cases until at least Grade 4;
- all teachers agree on the benefits of developing informal written methods as a bridge between mental and formal written methods;
- almost all teachers would still advocate the teaching of formal written methods (but at a later stage in the primary school than hitherto).

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From conjecture to proof...

Exploring patterns and relationships leading to 'proofs'

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Without proof there would be no mathematical knowledge, in a strict sense. It is the form of mathematical proof which sets Mathematics apart from the empirical sciences, and this significant difference should be made clear to students of mathematics.

In this workshop, Howard and Ken will engage participants in 'proof directioned' activities which are suitable for working with students at primary, middle and secondary school levels, and through which students can be acquiring the skills of developing and constructing proofs. The activities are located in 'Number' and involve discerning and exploring pattern, leading to the formulation of generalisations, conjectures and proofs.

Proof involves the formation of a chain of 'valid' reasoning that leads to a conclusion. It is a process of 'authentication' or a process wherein the truth or fallacy of a claim is established.

As a component of mathematical educational activity, proof sits in the following sequence of 'events', so to speak.

- (i) A mathematical situation is explored. The exploration may involve 'case studies' where calculations or computations are made, diagrams are drawn, discussions take place and the 'setting' is thoroughly and 'appropriately' examined.
- (ii) Arising from (i), patterns and relationships are discerned; students (and teacher) articulate belief that 'such and such' a pattern or relationship may well prevail throughout; that is, there is the initial formation of what may well become a 'conjecture' to be subjected to the rigours of proof.

- (iii) Following on from (ii), there is a stage where conviction concerning the possible ‘true existence’ of the ‘suspected’ patterns and relationships is ‘built and reinforced’.
- (iv) With ‘conviction and belief’ firmly established, a conjecture or hypothesis statement is made; such a statement claims that ‘such and such’ a pattern or relationship could be true.
- (v) And finally comes a proof itself. A proof provides ‘explanation’ of a pattern, relationship, or result believed, by the exploring students, to be possibly true.

In educational contexts, proofs may vary in their formality and rigour. On the one hand, a series of diagrams, calculated or algebraic ‘case studies’, accompanied by written or verbal supporting explanation may be acceptable proofs, while complete and totally mathematically rigorous argument presentation may be required, on the other hand. The nature of proof deemed to be acceptable is governed largely by the age and experience of the learner and the exploratory context involved. But whatever, students should come to accept the need to provide a proof, become aware of a variety of ‘proof forms’, and be able to follow the ‘substance and structure’ of a presented proof, while recognising any inherent limitations.

Over the past 5 years or so we have been developing and refining ‘proof directioned’ activities within the realm of Number. The activities stem from discerning structure and relationship within the infinite set of whole numbers, extending to integers and rational numbers. The thinking involved in observing and expressing structures and relationships with whole numbers is, indeed, algebraic in form. Certainly, when the aspect of generalisation of such observed structures and relationships is considered across the whole numbers, algebraic thinking is accompanied by ‘algebraic language’ which is characterised by the use of ‘pronumerals’. In a real sense, these experiences and actions are an essential part of learning what algebra is, in the form of ‘generalised arithmetic’. Additionally, the process of developing proofs sees the algebra as a very useful mathematical tool.

In many respects, the work with which we have been involved is concerned with these ‘inseparable’ aspects of mathematical education:

- the teaching and learning of algebra
- the fostering of a spirit of ‘mathematical exploration and discovery’
- the construction of a ‘mathematical proof’.

In the workshop, we group activities to focus on developing the above facets of mathematical education with primary, middle and secondary school students. The broad categories are:

- (i) consecutiveness, wherein we consider whole numbers, multiples of whole numbers, some properties of, and relationships concerning, consecutive numbers;
- (ii) square numbers, where we investigate some relationships involving square numbers;

- (iii) some patterns and relationships on hundreds number charts, calendars and multiplication grids.

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The hassle-free way to give your lessons that technological edge

Bernie O'Sullivan

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The State Review Panel (Queensland) Report to Schools 2001 stated:

It is disappointing that many schools have not employed the full capabilities of spreadsheeting software which can easily lead students into graphing with all its associated statistical and presentation capabilities.

More and more our students are becoming technologically literate, often more so than ourselves. As this trend continues, we will be frequently confronted with an expectation that the presentation of our subject should reflect the technology widely available in society. The drawback? Resources, both in time and money.

Microsoft *Excel* provides a ready made solution. It is software that most home (and school) users already have, and with the files I have been creating, teachers have at their fingertips what they need to commence turning their classrooms into technological centres.

We will cover topics such as:

- how to attach scrollbars to spreadsheets, enabling us to change any pro-numeral with the click of a mouse, thus investigating changes produced in families of functions and on graphs;
- simple 'programming' tips that can be inserted into cells;
- how to assign macros to a worksheet to automate tasks;
- tips for linking your graphing calculator and PC and much more.

Visit <http://www.schoolyard.stlukes.net.au/> and follow the links to the Maths page, there you will find spreadsheets that can be used to demonstrate and animate topics such as differential calculus, statistics, families of functions and more.

This workshop will help the novice user reach a level of proficiency where they will feel comfortable with, and capable of, introducing technology frequently into their classrooms.

Looking inside classrooms: Teachers engaging students, students being responsive

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Teachers engage students through activities, equipment, physical involvement, challenge, and discussion. Students respond as a result of their thinking visually, conceptually, strategically, and through meta-thinking, feelings, beliefs and attitudes. Based on classroom visits, key teaching aspects became themes of the analysis. These are best practice examples worthy of sharing with other teachers.

Inside classrooms

Over the past three years, I have had the privilege of observing some good teachers in practice. The number is over thirty and in most cases I was able to observe more than one lesson. These teachers work in different states of Australian and overseas where classroom conditions may not be quite as good as in an Australian classroom. Most classrooms were full of students from a variety of language backgrounds and for whom English was a second language. The schools themselves were both private and government schools. Each teacher was also interviewed about his or her teaching. Most of the lessons were on Space and Geometry. In addition, I have been able to read some written reports by teachers of their implementation of different lessons.

When observing these teachers, key issues kept arising:

- the teachers' background, personality, planning, risk-taking, and satisfaction with their teaching;
- the materials from which they were working;
- the selection of key activities which were adapted across a range of levels;
- the on-task behaviour of the students and how the teachers were achieving this;
- the interaction of the students in the classroom and their freedom to express ideas;
- the teachers' perceived roles within the school due partly to the school valuing quality teaching and inter-teacher cooperation.

There was diversity among the teachers and at times it can be said that they were restricted in achieving the standards they might have wished for one reason or another but this did not seem to dampen their enthusiasm for teaching.

The teachers

Undergraduate and postgraduate education

Three teachers referred specifically to their own engagement during initial teacher education or postservice courses. In two cases, the learning clearly involved the teachers trying out ideas like group activities, keeping records of student progress, and evaluating their own teaching. In the other case, the teacher simply said the lecturer was inspiring. Each had pursued these ideas. In two cases, this was despite the dominance of a textbook and a school culture that did not specifically encourage group work or concrete materials. Other teachers were self-engaged as a result of reflective practice and team sharing. A couple of teachers referred to having to fall back on their own resourcefulness and make up many teaching ideas on the small glimmer of ideas they had experienced during training or seen in other classrooms.

Teamwork and risk-taking

A few teachers were not specifically strong at teaching mathematics, they said, but with the encouragement of team work in the school, and building confidence with teaching another subject like English, they were planning lessons from which students were still likely to learn because the lessons were open-ended investigations and teachers could often learn with the students. Indeed, even where teachers were fairly confident and capable in mathematics, the ability to learn with the students and to take risks was very evident. The risk-taking included running a new activity, trying something which might result in a student saying something they may not have the answer for, and having a reasonable idea of the result but not being able to actually carry out the activity themselves in advance. Whole class discussion in which the teachers did not necessarily know all the answers nor give the answers were quite a feature of the classrooms. Getting the answers was not the major point of mathematics. Processes and conceptual understanding were far more important. For example, in a Year 6 classroom, one teacher was working on decimal fractions and she had cut and pasted some exercises from some textbooks. One of the questions was 0.5×0.8 . The answer did not give two decimal places as the teacher had suggested. Now the students were able to explain that these were both tenths and so the answer would be hundredths, 2 decimal places. However, they were getting the answer 0.4 with one decimal place. They checked the idea of a half of 0.8. Some girls checked with a calculator and explained. The teacher conceded a simple rule of 2 decimal places giving 2 decimal places was not quite clear.

Planned steps

Every quality lesson had several steps that the teachers had planned. There was generally an introduction in which key ideas and strategies were discussed with the students before they were sent off to small groups or pairs or individual work. The work

which the students had to do varied from a game, activity, drawing, or construction to book work. The book work might have been guided by a textbook or a worksheet but generally required some problem solving on the students' part. The book work might have consisted of recording during the activity or afterwards as a reflection or an application. Whenever there was activity, the teacher would know exactly what she or he wanted the students to learn, why the activity might be set with specific developmental steps, and what kind of questions would draw out the key ideas. Questions were asked of the students as they carried out the activity. For example, in one lesson using geoboards, the teacher had students first make one square and the whole class discussed these and why they were all squares. Then the class had to make many rectangles, followed by any four-sided shapes which could be sketched. These were then discussed in detail. All teachers planned some form of sharing session at the end of the lesson. For example, in one lesson using geoboards, the teacher had students first make one square and the whole class discussed these and why they were all squares. Then the class had to make many rectangles, followed by any four-sided shapes which could be sketched. These were then discussed in detail.

A particular feature of the classrooms was the use of paired tasks and active involvement of the students in the introductory phase. For example, in one Kindergarten class the students were each given a card and they had to find their partner and sit down. This led to a discussion which was the lead into the next part of the explanation. In this classroom the cards were dot and numeral cards. In a K-1 classroom, the cards had two and low three digit numbers. The students found partners for all sorts of reasons. For some they were the consecutive numbers, for others they were both even or part of counting by 5, some were the same digits, and so on. In a Year 3 classroom, the students were grouping a row of students into equal groups. In a Year 5 classroom, the students demonstrated pacing out steps across the front of the room, measuring the step in cm and then deciding the length in cm. They were then to apply this to some outdoor measurement around the school and to some map marking.

Realistic experiences

Even in the classrooms in which the teacher lacked resources or a good teachers' guide, the teachers planned as best they could building on a key idea to create a realistic mathematical experiences for the students. Among the various examples are drawing a map of their house, playing a game where two students at the front answered questions from the class on the topic, hunting for shapes in the environment, specific work for different ability groups, and stories. For example, odd and even were introduced by a story about these two people who liked to be visited by the numbers that were the same kind. In another classroom, an analogy or story about surfing reminded students of the procedural steps in a written algorithm.

Enjoyment

For one teacher, the challenge game was less competitive for the weaker students because the teacher rotated partners in both directions around the circle. There was more chance for two students to be more equally matched rather than the best student dominating. When a good student had a turn, they were really going to try to beat this

student. With many facial expressions and chatter, the teacher just made the whole thing enjoyable. If you were a good student, being challenged was a key aim and part of the fun. Another favourite game was Guess My Number. Since this was a Kindergarten-Year 1 class, the students had to guess the number in the teacher's head. She enjoyed this and it encouraged students to listen to each others' questions. Many students participated and she was able to involve the weaker students by letting them ask a question early. Another teacher also smiled the whole way through the class introductions as she involved different students in the explaining and examples. Everyone felt good about what they contributed. In another classroom the Support Teacher Learning Difficulties who was supporting the classroom teacher, frequently interacted with the teacher and involved the students in the challenges and conversations (almost banter) about the mathematics. These examples could be repeated for many classrooms. The encouragement to enjoy mathematics and to do it well were key goals emanating from the teachers and taken on board by the students.

Teacher satisfaction

All teachers appeared to put an effort into their teaching. They would prepare materials for the lessons, discuss them with colleagues, and when planning look up resource books and refer to the syllabus or teachers' guides. They also built up good resources for teaching and were able to use and adapt them over a number of years. They made an effort and they were satisfied with the results. They felt sure that their good rapport with the students and improvements in class discussions or other means of assessment were satisfying. This did not mean to say that they were satisfied about everything in the school. Some could point to issues with the school that disappointed them or the number of tasks they were asked to do outside the classroom.

Teacher support materials

Many of the teachers that I observed had been implementing *Count Me In Too*, *Count Me Into Space* or *Counting-On* programs. They generally commented favourably on the valuable theoretical background that these programs provided and how this assisted them to appreciate how students learn specific topics, to know how to teach certain topics, assess student development and prepare lessons that would benefit the students both individually and as a group. They particularly appreciated the wealth of lesson activities and ideas and, interestingly, the detailed lesson plans that were provided in the kits or illustrated on the videotapes.

In the overseas and interstate schools, the teachers did have some form of teacher's guide or textbook that encouraged group activity, hands-on experiences, and some kind of written work for students to complete to consolidate their knowledge. A particular point of interest was that none of the materials encouraged long sets of exercises. There was plenty of variety and different examples of the concept being discussed. Lessons would build on and provide experiences relevant to the concept being developed over a number of days. Discussion of the conceptual ideas was expected through the activity in small groups and as a whole class with the teacher.

For some of the teachers, they continued to rely on their teacher education, previous experiences, and to build up their own resources by making up their own activities and games and written exercises. Often this was done as a team of teachers. Some activities were favourites. For example, one Year 5 teacher had groups make body skeletons out of streamer paper by putting the paper around or along various parts of their body. However, the teacher allowed groups to decide whether they would make the skeleton full size or half size or some other ratio or fraction. This choice for the groups, plus allocation of different roles encouraged students' ownership. The teacher had used this activity from Year 2 to Year 6 simplifying it to suit the grade. In this case, he was keen to have students making fractions of the full length. One teacher adapted the 'find a partner' activity across a range of classes, introducing new concepts like even and odd numbers, and having larger numbers. Another favourite activity for this teacher was the challenge game. This was adapted, for example, to recognising the number of dots on a die, to adding the dots on two dice, adding a numeral and dots, making a two digit number by multiplying the numeral by ten and adding the other die's dots and so on. Another teacher had a variety of number jigsaws. These included the hundreds chart, a snakes and ladders board, and ten-frame numeral representations. Other favourite activities involved game boards with chance cards that were answered whenever certain squares were landed on. The cards were varied depending on the topic. Of course, bingo too was readily adapted to a range of topics and usually played in small groups with one of the students reading out the questions. One teacher commented that one particular kit was particularly useful because it supplied the materials. This gave the teacher a good start to collecting activities.

In Australian schools, the classrooms generally had a computer. However, during group activities this may or may not have been used. It seemed that a lack of appropriate software was a major hindrance to its use. Where teachers had software, a small group would work on the computers. The students were always working in pairs. Similarly calculators were rarely used, simply because the teachers did not know how to make the best use of them. Where teachers had observed, for example, the videotape *Young Children Using Calculators*, the teachers were more likely to use them. At other times they were used 'when the textbook said it was a calculator activity or exercise.' However, in a number of the classrooms where teachers were using best practice, the calculator was available and students selected when they would use it. These same students usually told me that they would prefer to work out the calculation in their head.

Student engagement

A most noticeable feature of all the classrooms that were visited, was the very high proportion of time and the very high proportion of students engaged in the class discussions, activities, and written work. This was particularly encouraging considering that three of the teachers told me they were given difficult classes because the principal needed the classes to improve in their behaviour. The key to success in some classes was an improvement in student literacy. Once students achieved a sense of satisfaction in being able to read and write, their interest in school improved. For some students, it was the encouragement to speak in the classroom that mattered.

Both the teachers and the students expected mathematics to be fun, hands-on, and challenging. Students were not expected to copy teachers' examples but to take ownership of their learning. This was achieved by encouraging students to make or draw their own objects or shapes or sums.

Another aspect of the classroom was the expected behaviour of the students by the teacher and hence the students. The teachers did not tolerate slack students. This does not mean that they were impatient with slow learners or students with learning or behaviour problems. These teachers handled these students particularly well. Teachers used a variety of devices from a 'singing crocodile' that timed how long students had to pack up, ticks for good or poor behaviour on the board, much praise and enthusiasm for student participation and attempts, graded materials and activities, paired and group work, and plenty of work. Explanations about the activity procedures were clearly made so even open investigations were carefully explained. Games and activities were frequently repeated but often with slight changes that made them more difficult. This meant that knowing the rules allowed students to use or develop the concepts better. Similarly class routines were well established. Materials were well organised. In multigrade classrooms, the teacher required students to work on their materials without noise. This was achieved by careful explanation and the type of work selected for students to work on without teacher intervention.

Teacher experience and preparation were key to successful on-task behaviour. This was the case even when the teachers had only been teaching for two or three years. These teachers were reflective about their practice and had quickly developed strategies that worked. They were constantly aware of the need to adapt and improve in this area with different students.

Student interaction

In all classrooms, there was extensive interaction. This was frequently with paired or group work and with whole class discussions. However, the classrooms were never disturbingly noisy, students spoke quietly except occasionally when excited. One teacher worked in a school environment where students were expected to work more as individuals but she had achieved some success with disruptive students and so had gained support from senior teachers. In other schools, some teachers were gradually convincing others of this approach. This was done mainly by team-teaching. At other times, teams had achieved a similar result by sharing preparation. The teachers realised that they needed to have good preparation for groups to work. In one school mathematics was always after a recess so the teacher could get the mathematics equipment ready. This was particularly necessary for measurement lessons but also for many others where games, arithmetic materials, papers and scissors, blocks and geoboards were required for space mathematics.

School environment

Most but not all the teachers were part of a Year team that worked on ideas for lessons, shared the organising and preparation of materials, and occasionally had the opportunity to team teach. In one school, this was a common practice for the whole

year. In other cases, the facilitator of new *Count Me In* programs worked with the class teachers to introduce ideas. Others were able to work with support teachers of one kind or another. However, there were also classrooms where the teachers were more isolated in their work. Some regular school inservices were held in these schools but there was less opportunity for the teacher who was not so keen on mathematics to improve. Generally these teachers did not volunteer to be observed during school visits. It was noticeable that most of the schools had a strong collegiality among the staff. The hard work that the teachers put into the care and discipline of the classes often required this kind of support. Teachers were supported and encouraged at introducing new challenging programs like debating. Casual staff were teachers who regularly had work at the school. The support of the principal or senior executive in encouraging textbooks as resources only, group work, government-sponsored pilot projects and research, and new equipment was significant. When mathematics was on the agenda for the school, there was increased enthusiasm, risk-taking, and a sense of achievement for both the teachers and their students.

Students' responsiveness

A young Kindergarten student was generally struggling with counting and recognising numerals but even in challenge activities, this child had the biggest smile on her face. She and the teacher were enjoying every minute of her part in participating in class discussions and class games. The key was that she had a go. She was responsive and the teacher was always able to use her answer and give her encouragement and direct her thinking. The episodes were not long so the rest of the class listened until they were challenged. Every class that was visited had some students like this one. The only time I saw students feeling ashamed was when the class had a competitive game, even though the majority of the class enjoyed it. Interestingly, at least all the teachers who had this time of activity had incorporated procedures to reduce the competitiveness between students.

In another activity, the students had to measure different jumps and compare them, they worked in threes, knew exactly what they had to do, they did it cooperatively and recorded their results including who had the biggest jump. After some class discussion, every student wrote about the mathematics they learnt in their journal. The experiences were being translated into verbal communications.

In another classroom, the students were putting together the hundreds chart jigsaw. This was surprisingly challenging to the groups. They all had turns and would talk about the strategies that they could use. It was interesting that most started at the bottom of the jigsaw although there was no explanation for this. This seemed to make the task harder and so students might then start further up the puzzle. The position of the pieces influenced their thinking. They needed to have both a visual idea of the chart but also conceptual understanding was applied to complete the array. They were certainly consolidating the pattern of the chart and the 10s and ones of the numbers. The more students were challenged, the more they seemed to respond positively to the climate. One exception was a student in a speed activity who really needed teacher approval and somewhat cheated to finish second in the class. His answers were quite inaccurate and his response was a little of shame. The teacher on a later occasion

discussed the issue with him rather than bring it up at that point in time with the class. Nevertheless, this student listened to the other students as they shared their various strategies for completing the table. Such strategies were highlighting the patterns in multiplication tables.

The fascinating discussion described above about the number of places in a decimal fraction illustrated how the class environment would influence individual's thinking. They were sharing their thinking and in so doing developing their ideas. When the two girls felt the answer only made sense if it were hundredths and that 0.4 was another way of writing 0.40, they not only checked their thinking with the calculator but then proceeded to explain to the teacher what had happened and the reason for it. It was clear they wanted the reassurance from the calculator but they could be a means of checking and they also knew what was happening. The class listened to the various responses to the question and the explanations.

Conclusion

Classrooms and their school environment are key factors for the learning of mathematics. Enthusiastic and effective teachers of mathematics try out new ideas with sufficient support from professional development materials, from their own background experience in organising activities and developing sound classroom discussions, and from team teaching or other forms of reflective professional development. These teachers monitor the progress of their students and provide realistic lessons. The students are responsive in the classroom. They enjoy mathematics and treat a challenge with thought. Their thinking is influenced by the materials they are using, their peers and teacher, the problem or game that is set. Their thinking leads to their responses. They may be drawing upon visual imagery, concepts, strategies, self-checking, and feelings in order to be responsive and participatory in the classroom. There was no doubt that the group work and activities had a significant impact on students being responsive, working with the group and then coming up with the next step or piece of conceptual understanding.

The development of mathematical strategies in the early years

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Current research has shown that children develop a repertoire of mathematical strategies by progressing through five Counting Stages. The importance for classroom teachers to be able to identify each child's strategies and thus their counting stage will be stressed as a starting point for mathematics/numeracy teaching in the early years. The results of clinical interviews used for assessing each child's Counting Stage will be discussed. This paper will highlight the strategies that have been used to assist children overcome common difficulties. These strategies can be incorporated by classroom teachers to enhance their mathematics program so all students succeed with mathematical tasks.

Introduction

In this workshop we will highlight current research that has shown children develop a repertoire of mathematical strategies by progressing through five *counting stages*. The importance for classroom teachers to be able to identify each child's strategies and thus their counting stage will be stressed as a starting point for mathematics/numeracy teaching in the early years. The results of clinical interviews used for assessing each child's counting stage will be discussed. This paper will highlight the strategies that have been used to assist children overcome common difficulties. These strategies can be incorporated by classroom teachers to enhance their mathematics program so all students succeed with mathematical tasks.

Current research about children's early arithmetical learning documents students' progression through the counting stages (Steffe, von Glasersfeld, Richards & Cobb, 1983; 1988; Wright, 1991; 1996) and are summarised below.

1. Perceptual

Students are limited to counting those items they can perceive. For example, if asked to count a collection they will attempt to count any items they can see, hear or feel but would be unable to count any hidden items.

2. Figurative

Students count from one when solving addition problems with screened collections. They appear to visualise the items and all movements are important. (Often typified by the hand waving over hidden objects.) If required to add two collections of six and three the student must first count the six items to understand the meaning of 'six', then count the three items, then count the whole collection of six and three.

3. Initial number sequence

Students can now count on to solve addition and missing addend problems with screened collections. They no longer count from one but begin from the appropriate number. If adding two collections of six and three, students commence the count at six and then count on: six, seven, eight, nine.

4. Implicitly nested number sequence

Students are able to focus on the collection of unit items as one thing, as well as the abstract unit items. They can count-on and count-down, choosing the most appropriate to solve problems. They generally count down to solve subtraction.

5. Explicitly nested number sequence

Students are simultaneously aware of two number sequences and can disembed smaller composite units from the composite unit that contains it, and then compare them. They understand that addition and subtraction are inverse operations.

Clinical interviews

If teachers are to focus on children's strategies they need to engage in a one-to-one interview where a child is required to perform tasks in front of, and with prompting and probing from, an interviewer. The clinical interview is among the most widely used approaches of today (Steffe, 1991). The clinical interview supplies 'in-depth' information on which to construe student's thinking and cognitive processing. The questions asked and the observations made using task-based clinical interviews depend on the theory the teacher/interviewer brings to it. Using a clinical interview yields information that is not easily accessible from other sources such as paper-and-pencil tests. The primary goal of interviewing is to explore the limits of the student's thinking. The process of thinking is considered more important than the correct solution. Student's verbal and non-verbal behaviour is observed and from the observations the interviewer infers something about the student's internal representations, thought processes, problem-solving methods, or mathematical understandings. A series of sessions are likely to be needed for the interviewer to develop and test his or her model of the student's understanding.

Mathematics intervention

Developed in 1993 the *Mathematics Intervention* program (Pearn & Merrifield, 1996) featured elements of both Reading Recovery (Clay, 1987) and *Mathematics Recovery* (Wright, 1991; 1996) and offered students the chance to experience success in mathematics by developing the basic concepts of number upon which they build their understanding of mathematics. Students are withdrawn from their classes and work in small groups with the additional assistance being provided by a trained specialist teacher who provides a program that promotes the development of their mathematical skills and more efficient strategies.

An instrument to determine students' counting stages was developed, administered and consequently modified by three teachers. This is called the *Initial Clinical Assessment Procedure – Mathematics (ICAPM), Level AA* (Pearn, Merrifield, Mihalic, & Hunting, 1994). The focus is on number although we acknowledge the importance of a breadth of mathematical activities. However, as the *Curriculum and Standards Framework [CSF]* states:

As a student acquires an appreciation of different levels of understanding of number, intersections occur with other mathematical studies in ways which give number a central unifying role. Work in the number strand links with work in all other strands... Later work in all strands requires that they understand and work confidently with all kinds of numbers (Board of Studies, 1995, p. 42).

The clinical interview included tasks that ascertained the facility of the children's verbal counting skills, their knowledge of the number word sequence and tasks that would help ascertain their counting stage level. The clinical interview takes ten minutes and includes verbal counting tasks such as:

‘Can you count out loud for me, beginning at one, until I tell you to stop?’

‘What number comes *after* 4?’

‘What number comes *before* 15?’

There were only two tasks based on the counting stages. The first counting stage task was designed to determine whether the child can count-on.

(Ten counters are displayed.)

‘Here are some counters. Count them.’

(Cover all the counters, remove two and display.)

‘How many counters are under the paper?’

By carefully observing the children's solution methods, interviewers ensured that they were aware of the strategies being used and if needed the following prompts were given: ‘How did you work that out?’ or ‘How did you do that?’ For example, Table 1 highlights the typical responses of children and their corresponding counting stages to a task from the initial interview given to all Year 1 students.

Table 1: Typical responses to counting stage task.

Counting Stage	Task:
	'There are six counters on the table. Under this paper are three counters.' (Lift paper briefly). 'How many counters do I have altogether?'
0	Response is usually a guess. Children attempt to recite verbal sequence but have no one-to one correspondence.
1	Children can only count things they can see, hear or feel so guess responses when given tasks with some counters that are covered.
2	Usually incorrect response. Children count all collections from one. For example, will count the six counters starting at one, count the three counters starting at one, then attempt to count the nine counters again starting at one.
3	Children count on from the first collection of six: six, seven, eight, nine.
4 & 5	Children give instant response and can justify it: 'I know $3 + 3 = 6$ and another 3 is 9'.

Most Year 1 children were successful in counting forwards by ones to 20 and backwards by ones from ten, counted patterns of dots and counted out exactly 14 beads. They were less successful identifying the numbers between the numbers six and twelve or determining numbers 'before' or 'after' a given number. For example, of a sample of 238 Year 1 students interviewed:

- 40% were unable to count backwards from 20 to one by ones;
- 62% had difficulty counting by twos from 2 to 24, 39 % with fives from 5 to 60;
- 36% were unable to count by tens from 10 to 100;
- 38% could not say the numbers 'between' six and twelve;
- 23% confused 'before' and 'after';
- 14% could not count out exactly 14 counters;
- 17% were unable to accurately count the patterns of dots on a card;
- 59% could not correctly name the numerals 13, 14, 15, 31, 41, 51 (most confused 31 with 13 or 30, 41 with 14 or 40 and 51 with 15 or 50);
- 24% were only able to count things they could see, hear or feel (counting stage 1);
- 38% were unable to 'count back' or 'down to' thus had difficulty with subtraction (counting stage 3).

Common difficulties

Since the development of mathematics intervention in 1993 common problems were exhibited by young children considered to be mathematically 'at risk'. These difficulties include the following:

1. Difficulties in elaborating the number sequence.
 - a) This could be an incorrect count where the right words are used but in the wrong order: one, two, three, four, six, five.
 - b) A number could be omitted: one, two, three, five, six.
 - c) Sometimes children confuse two sequences as in: ten, eleven, twelve, thirty, forty, fifty, sixty, seventy, eighty, ninety, twenty.
2. Children exhibit little or no one-to-one correspondence. Children 'at risk' have difficulty in coordinating their spoken number sequence with the physical counting of objects.
3. Once children are able to count past ten there is confusion with the '-teen' words and '-ty' words. Incorrect sequences seem to result from the irregularities in the English system of words for the words from ten to twenty and the decade words.
4. Children experiencing difficulty learning the forward counting sequence to 20 have great difficulty in counting backwards from 20.
5. Bridging both forwards and backwards. Several children did not recognise that 20 came after 19, and similarly 30 after 29. Each bridging of the decades appeared to be an additional hurdle for the child.
6. Difficulties in understanding what the symbols mean. Although children had encountered the signs: +, -, = in their classroom mathematics they do not appear to understand what they mean.

Discussion

Experience with interviewing children in the early years leads us to believe that there is an earlier counting stage: Stage 0. This is the stage where children recite the verbal sequence, either successfully or unsuccessfully, but do not seem to realise that we count for a purpose. Thus their counting is really just a recitation of number names. At this stage they do not use one-to-one correspondence, that is, they are unable to co-ordinate the number words with the items being counted.

Within the counting stages there appears to be a progression in the types of responses by children at the beginning of the stage compared to a more sophisticated or efficient strategy when they have consolidated within that stage. In Table 2, I have given an example from Counting Stage 3, an example of the task and the responses that indicate whether a child is beginning at that stage or is using a more efficient strategy within that counting task. For example, Stage 3 the '*count on*' stage has shown how students'

responses become more efficient when they count on from the larger number rather than the first number they are given.

Table 2: Counting Stages and indicators showing progression in strategies.

Counting stage	Example of task	Indicator for 'beginning'	Indicator for 'efficiency'
Stage 3: <i>Count on</i> . Students begin from the appropriate number to solve addition and missing addend problems with screened collections.	There are three counters on the table. 'Under this paper are seven counters.'(Lift paper briefly) 'How many counters do I have altogether?'	Student counts on from the three counters he/she can see: 3 4 5 6 7 8 9 10. That is, they count on from the first number given.	Student counts on from the larger number: 7 8 9 10. That is, they count on from the most efficient number.

Teaching strategies

Several teaching strategies that will allow children to experience success with mathematics in the early years of schooling include:

Verbal counting

To facilitate the improvement of children's counting skills time must be spent each lesson counting both orally and with structured materials. For example, counting beads on a bead frame, collections of counters, beads, toys and in fact anything countable. Emphasis is also placed on the pronunciation of the number words. Every year, *mathematics intervention* teachers have observed that children experience difficulties with the number sequence due to poor speech especially with the '-teen' and '-ty' words. Quite frequently the mispronunciation had been missed by classroom teachers.

Questioning

Teachers need to be skilled in questioning and able to ask mathematical questions using the correct mathematical language. Skilful questioning by the teacher is imperative to ensure that the children's mathematical knowledge can be used to form a strong foundation on which to build further mathematical knowledge. Children should be expected to explain their strategies to both the teacher and other students and where necessary prompts should be given such as, 'How did you do that?'

Alternative solutions

Children are encouraged to think of, and discuss, different ways tasks could be solved. Teachers must refrain from saying whether answer is correct or incorrect or that one procedure is better than another. Teacher should encourage children to explain their solutions and to tell each other whether or not an explanation makes sense to them.

Invented strategies

Carpenter et al. (1998) noted that many children constructed their own procedures for adding and subtracting multi-digit numbers without using concrete materials. They called these strategies 'invented strategies'. They found that children who use invented strategies developed knowledge of base-ten number concepts earlier than children who relied more on standard algorithms. In their study they found that children made 'relatively few conceptual errors in using invented strategies, whereas children exhibited a number of buggy procedures in using standard algorithms' (p. 19). Children need to be encouraged to develop and record their own strategies, first with and then concrete materials.

Games

To ensure active participation in mathematics, games are used wherever appropriate. The variety of the games depended on the imagination and skill of the teachers. Games using dice are used to compare numbers, add and subtract numbers and to make up their own sums. It is this ownership of the mathematics that becomes a very powerful tool in learning. Different sized dice can be used depending on the child's ability. A game called *Twenty* was devised to assist children to make the transition from counting all the counters (Stage 1 and 2) to counting on (Stage 3), or 'counting back' which are much more powerful strategies. Children need to use more efficient strategies if they are to succeed with the formal processes of addition and subtraction.

Conclusion

This research has highlighted the differences in children's mathematical knowledge and the type of whole number strategies they use when solving tasks set in different contexts. Year 1 children who were successful with the tasks from the initial interviews counted fluently by ones, twos, fives and tens from a given number, and demonstrated their ability to choose and use the appropriate strategy of count on and count back. However, some Year 1 children experiencing difficulties with the verbal counting sequence and were at either Counting Stage 0, or 1 or 2. That is, they were unable to count successfully or used the strategy of 'count-all'. When unsure of an answer these children guessed with no attempt to confirm their answer.

By being aware of the child's mathematical knowledge and the types of strategies used the teacher is able to design appropriate activities to extend his/her mathematical understanding. Maybe then most children would be able to 'acquire mathematical skills and knowledge so that they can deal confidently and competently with daily life' (Board of Studies, 1995, p. 9). Maybe then mathematics at school would be 'a positive experience in which students develop confidence and a sense of achievement from what they learn' (Board of Studies, 1995, p. 9). The question remains: Can we help teachers to provide mathematics activities in the early years so that children are able to develop their own strategies that will allow them to be less reliant on inefficient rules and procedures?

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Fractions: Using the measurement model to develop understanding

Catherine Pearn

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Several researchers have noted how children's whole number schemes can interfere with their efforts to learn fractions. An Australian study found that children who were successful with the solution of the rational number tasks exhibited greater whole number knowledge and more flexible solution strategies. Behr and Post (1988) indicated that children needed to be competent in the four operations of whole numbers, along with an understanding of measurement, to enable them to understand rational numbers. This session will be a 'hands on' session that focuses on the use of paper folding, fraction walls and number lines to develop an understanding of fractions using a measurement model.

Introduction

Over the past twenty years, research on rational number learning has focussed on the development of basic fraction concepts, including partitioning of a whole into fractional parts, naming of fractional parts, and order and equivalence. Kieren (1976) distinguished seven interpretations of rational number which were necessary to enable the learner to acquire sound rational number knowledge, but subsequently condensed these into five: whole-part relations, ratios, quotients, measures and operators (Kieren, 1980; 1988). Kieren suggested that children have to develop the appropriate images, actions and language to precede the formal work with fractions. Saenz-Ludlow (1994) maintained that students need to conceptualise fractions as quantities before being introduced to standard fractional symbolic computational algorithms. Streefland (1984) discussed the importance of students constructing their own understanding of fractions by constructing the procedures of the operations, rules and language of fractions.

Behr and Post (1988) suggested that children needed to be competent in the four operations of whole numbers, along with an understanding of measurement, to enable them to understand rational numbers. They suggested that rational numbers are the first set of numbers experienced by children that are not dependent on a counting

algorithm. Steffe and Olive (1990, 1993) showed that concepts and operations represented by children's natural language are used in their construction of fraction knowledge. The goal of their project was to identify accommodations which children made to their counting sequences that would yield fraction schemes. Two distinct fraction schemes emerged from the research. In the iterative scheme, children established a unit fraction as part of a continuous but segmented unit. From this, children developed their own fraction knowledge by iterating unit fractions. The foundation of a measurement scheme occurred when the children's number sequence was modified to form a connected number sequence.

An Australian research project (Hunting, Davis & Pearn, 1996) was designed to investigate the extent to which children's thinking processes might be associated with qualitative differences in their whole number knowledge when solving rational number tasks. Twenty-eight Grade 3 children of average age 8 years 2 months from a predominantly middle class state primary school were interviewed with ten tasks designed to ascertain the children's understanding of both whole number and rational number concepts. This research revealed the vast difference in the children's mathematical knowledge and the type of whole number strategies they used. This analysis suggests that children who used a variety of strategies to solve whole number tasks are more successful, and use superior strategies, when solving rational number tasks. Children, who relied on rules and procedures to solve whole number tasks, were less successful with rational number tasks. They experienced some success with partitioning and ratio tasks but little or no success with fraction tasks set in various contexts. This study raises several questions about conventional approaches to teaching fractions. Most children studied had difficulty understanding the language of fractions. While most students were successful with tasks involving one-half, very few understood, or were successful, with tasks involving other unit fractions.

As there appears to be an inter-dependence between the development of rational number knowledge and whole number knowledge, fraction tasks need to be given that allow children to develop numerical relationships and strategies flexible enough to be used in various contexts. Success with the ratio tasks indicated that problems and tasks developed in the context of sharing discrete items would be a good starting point for the teaching of fractions in the early years of schooling with emphasis on the introduction of appropriate language.

This workshop focuses on the development of understanding using the measurement model as used as part of professional development in conjunction with the Success In Numeracy Education [SINE] project. SINE is the major numeracy approach being implemented in Victorian Catholic schools. SINE is designed to assist teachers to identify the mathematical understanding of the students they teach and to develop activities to help all students to progress at their relative level of understanding. Two components of SINE have been piloted and are now being implemented: SINE Prep to Year 4, and SINE Years 5 & 6. This workshop focuses on the professional development work with Fractions being undertaken as part of the pilot program for SINE Years 5 to 8 (Pearn, Stephens & Lewis, In press). This workshop focuses on the difficulties highlighted by both the Fraction Interview and Fraction Screening Tests A and B used in primary and secondary catholic schools in Victoria.

Using the measurement model

In this workshop I will be using a model that has already been used in many primary and secondary schools in Victoria. The emphasis of the workshop is on a hands-on approach to focus on the meaning of fractions as a part of a whole and using this to develop the understanding of a fraction as a number. This workshop is modelled as if being presented in a classroom and there is an emphasis on the use of appropriate and exploratory questioning. This model looks at the move from paper folding to Fraction Walls then to marking fractions on a number line. Time will also be given to asking teachers to translate fraction symbols into everyday English and vice versa to develop computational understanding. Teachers will then be shown examples of students' work and asked to decide what the work indicates about the students understanding about fractions. This session will start with teachers being shown the fraction two-thirds and asked to draw a picture or a diagram to show what this symbol means. Previous experience has shown that most teachers will draw a circle divided into three parts not necessarily equal. Teachers are then asked to draw a different representation of two-thirds.

Paper folding

Teachers are given a piece of paper streamer that is 20 cm long and asked to fold it into two equal pieces. Discussion includes questions such as:

- How many pieces are there?
- How many folds are there?
- What do we call each piece?
- How many halves are there in a whole?

Teachers are then asked to fold their paper streamer half in half. Before opening their paper they are asked:

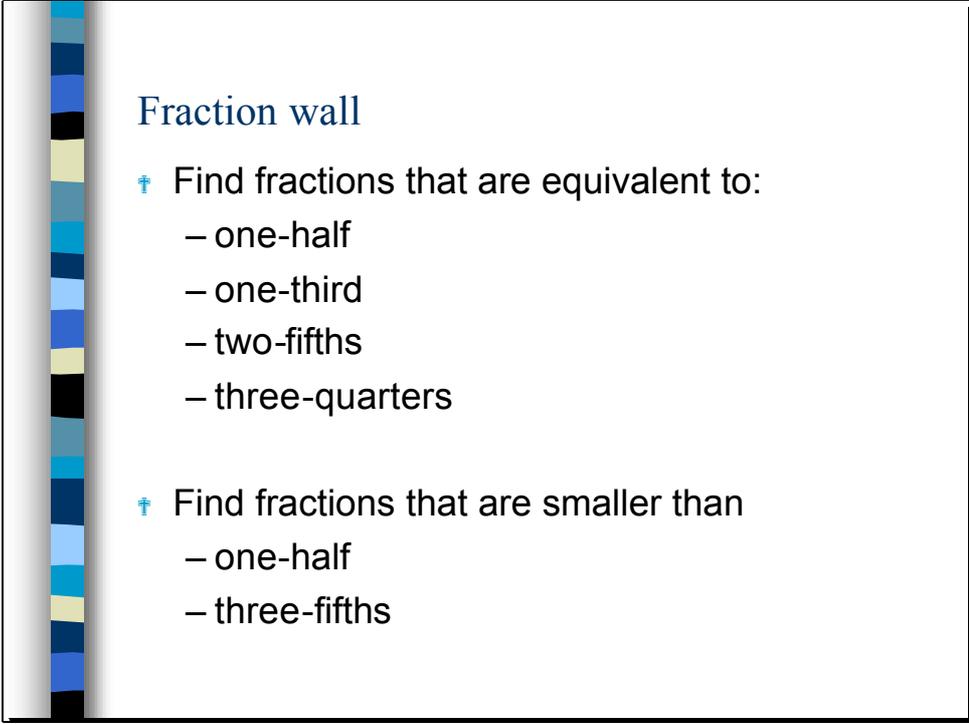
- How many pieces will there be?
- How many folds will there be?
- What do we call each piece?

Teachers are then asked to show: one-quarter, two-quarters, three-quarters, four-quarters, one-half and one whole. After folding their paper streamer in eighths teachers again will be asked questions that involve equivalence, showing fractions that are larger than..., fractions that are smaller than..., and questions such as, 'Show me a fraction that is larger than one-eighth but smaller than one-half?'

Another paper strip will be folded into halves, fifths, tenths and one into thirds, sixths, ninths and similar questions asked as for the halves, quarters and eighths. Teachers are challenged to fold a piece of streamer into sevenths.

In Figure 1 is an example of the *PowerPoint* slide that enables teachers to keep track of the instructions for the paper folding activity.

Using the fraction wall teachers will be asked similar questions to those asked using the paper strips that relate to equivalence: fractions smaller than..., fractions larger than... and fractions that are smaller than... but larger than... (see for example the *PowerPoint* slide in Figure 3 with some of the questions).



Fraction wall

† Find fractions that are equivalent to:

- one-half
- one-third
- two-fifths
- three-quarters

† Find fractions that are smaller than

- one-half
- three-fifths

Figure 3. PowerPoint slide with fraction wall tasks.

Circular models

Many worksheets and texts focus on students shading circular or rectangular shapes. Usually the divisions have been marked on these shapes and students are asked to shade the required number of pieces. Teachers will be asked to divide circular shapes without the use of compass or protractor which will highlight the difficulties experienced by students.

Number lines

Using either their paper folding or their fraction wall, teachers will be asked to mark several fractions on their number lines marked 0 to 1. After completing this task teachers will be given number lines marked 1 and 2; 5 and 10; 20 and 25; 0 and one-half. To complete this activity teachers will be chosen to place a fraction on a piece of rope marked 0, 1 and 2. They need to justify why they have placed their fractions in the place they choose.

Fractions as numbers

After completing this work on number lines teachers should be comfortable with the notion of fractions as numbers and this leads into work involving the language of fractions. Teachers will be asked to write in words (everyday English) some expressions such as:

$$\frac{1}{2} \times \frac{1}{2}$$

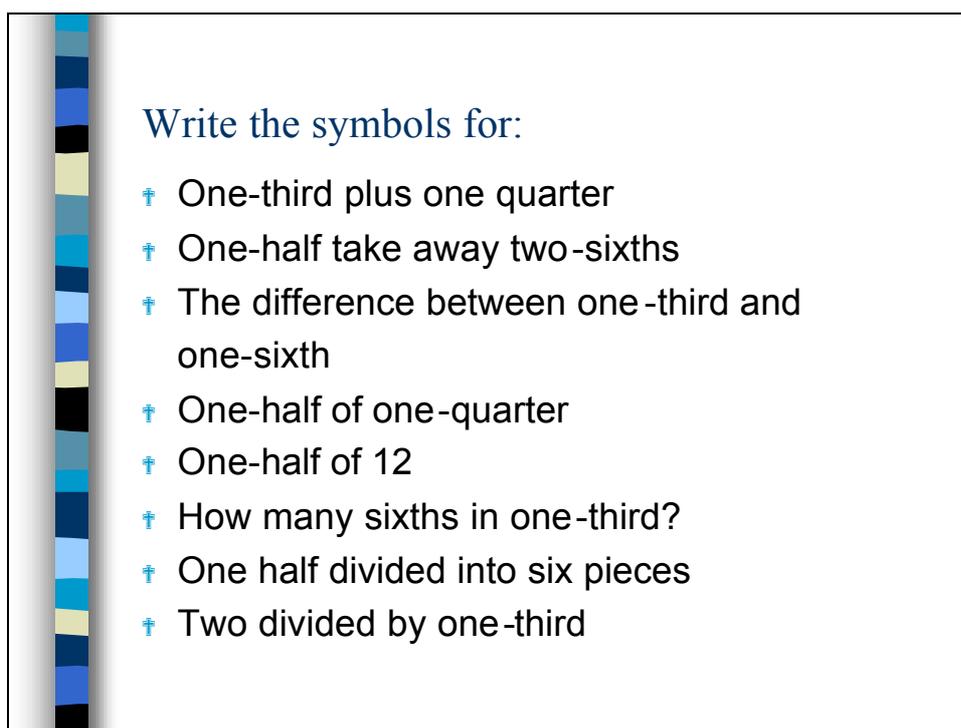
$$\frac{1}{2} \div \frac{1}{6}$$

and then asked to put into symbols questions such as:

- How many quarters in one-half?
- How many thirds in two wholes?

For example, see Figure 4.

Using number lines, teachers will be asked to complete a variety of calculations.



Write the symbols for:

- † One-third plus one quarter
- † One-half take away two-sixths
- † The difference between one-third and one-sixth
- † One-half of one-quarter
- † One-half of 12
- † How many sixths in one-third?
- † One half divided into six pieces
- † Two divided by one-third

Figure 4. PowerPoint slide with task asking teachers to convert words to symbols.

Analysis of student work

Teachers will be given examples of four Year 5–8 students' responses from Screening Test A (Pearn, Stephens & Lewis; 2002) and asked to comment on the following:

- What do these responses from the Fraction Screening Test reveal about the four students?
- Which tasks are the students successful with?
- Which tasks reveal misconceptions about students' thinking about fractions?
- What are some of the common misconceptions?
- What implications are there for teaching fractions in Years 5–8?

Conclusion

This series of activities based on the measurement model, using paper folding to complete the fraction wall to then developing the understanding of fractions as numbers is just one model that can be used to develop an understanding of fractions. This approach highlights the need to develop fractional language and the ability to read both words and fractional symbols.

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Partitioning — The missing link in building fraction knowledge and confidence

Dianne Siemon

RMIT University (Bundoora), Vic.

The Middle Years Numeracy Research Project (MYNRP), conducted in Victoria from November 1999 to November 2000, used relatively open-ended, 'rich assessment' tasks to measure the numeracy performance of approximately 7000 students in Years 5 to 9. The tasks value mathematical content knowledge as well as strategic and contextual knowledge and generally allow all learners to make a start.

For the purposes of the MYNRP, numeracy in the middle years was seen to involve core mathematical knowledge (in this case, number sense, measurement and data sense and spatial sense as elaborated in the *National Numeracy Benchmarks* for Years 5 and 7 (1997):

the capacity to critically apply what is known in a particular context to achieve a desired purpose; and the actual processes and strategies needed to communicate what was done and why.

Data from the final stage of the project indicates that teachers working in professional teams in a coordinated and purposeful way do make a difference to student numeracy outcomes, particularly where there was concerted focus on 'good' mathematics teaching. That is, the use of problem solving, extended discussion, student explanations, rich assessment and a range of materials, tasks and activities. However, the research also suggests that systems and schools still face a significant challenge in recognising and dealing with the issues involved in teaching and learning for numeracy at this level.

'Hotspots' identified by the research suggest that we need to pay careful attention to the 'big ideas' in mathematics and foster students' capacity to critically reflect on their learning. In particular, it would appear that we need to focus on the development of place-value, multiplicative thinking, rational number ideas, and what is needed to help students progress to the next 'big idea'. One of the objects of teaching and learning mathematics is to help the learner create meaningful mental objects that can be

manipulated, considered, and used flexibly and creatively to achieve some purpose. This requires that teachers are knowledgeable of developmental pathways and key learning trajectories, so that all students at all levels have the opportunity to learn the mathematics they need to progress to further study and effective, rewarding citizenship.

Why fractions?

It is no longer acceptable that students leave school without the foundation knowledge, skills and dispositions they need to be able to function effectively in modern society. This includes the ability to read, interpret and act upon a much larger range of texts than those encountered by previous generations. In an analysis of commonly encountered texts, that is, texts that at least one member of a household might need to, want to, or have to deal with on a daily, weekly, monthly or annual basis (see BLM1), approximately 90% were identified as requiring some degree of quantitative and/or spatial reasoning. Of these texts, the mathematical knowledge most commonly required was some understanding of rational number and proportional reasoning, that is, fractions, decimals, percent, ratio and proportion. An ability to deal with a wide range of texts requires more than literacy – it requires a genuine understanding of key underpinning ideas and a capacity to read, interpret and use a variety of symbolic, spatial and quantitative texts. This capacity is a core component of what it means to be numerate.

Formalising fraction ideas in the middle years

1. Review fraction language and ideas using continuous (e.g. chocolate bars, pizzas, fruit) and discrete fraction models (e.g. children in the grade, eggs in an egg-carton).

Continuous

e.g. 2 and 3 quarter pizzas

e.g. 2 thirds of the netball court

Discrete

e.g. half the grade to art, half to the library

e.g. 2 out of 12 eggs are cracked

2. Practice naming and recording every-day fractions using oral and written language, distinguishing between the count (how many) and the part (how much)

e.g. 3 fifths, 3 out of 5 equal parts;

2 wholes and 3 quarters (fourths)

3. Use practical examples and non-examples to ensure foundation ideas are in place, that is,

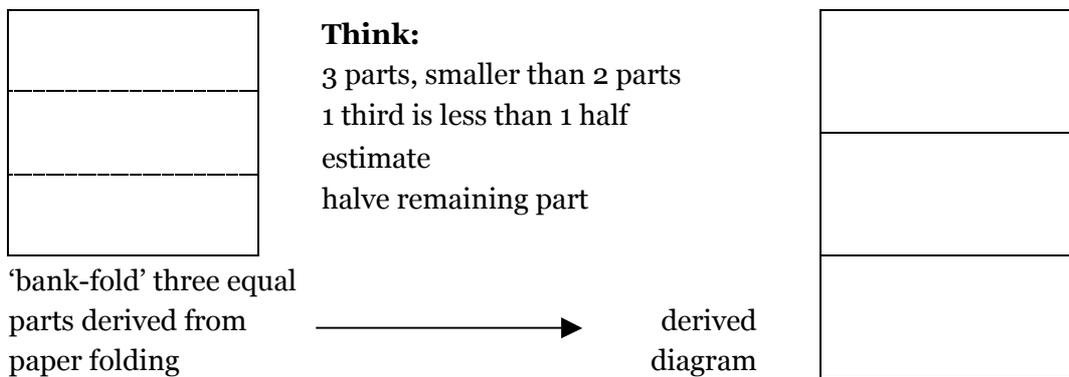
- an appreciation of part-whole relationships and the requirement for equal parts,

- recognition of the relationship between the number of equal parts and the name of the parts (denominator idea), and
- an understanding of how fractions are counted (numerator idea)

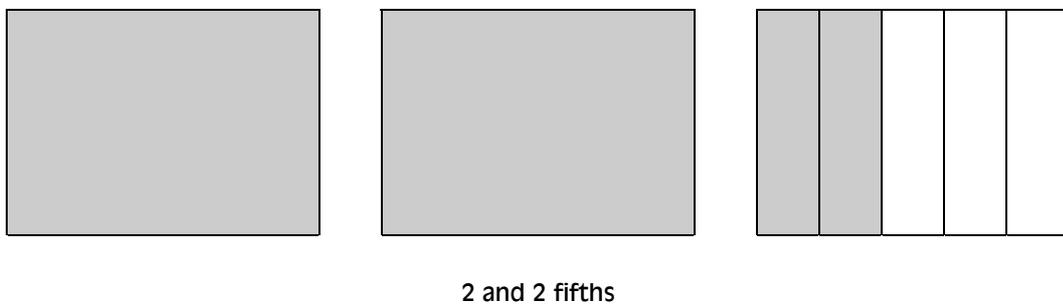
4. Introduce the ‘missing link’ — PARTITIONING — to support the making and naming of simple common fractions and an awareness that the larger the number of parts, the smaller they are.

Use ‘kindergarten squares’ to investigate ‘halving’. Explore and teach strategies for ‘thirding’ and ‘fifthing’ derived from paper folding/rope experiments and estimation based on reasoning about the size of the parts.

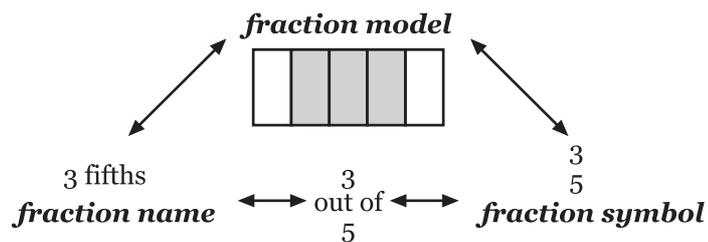
e.g. Use kindergarten squares (coloured paper) to make and name fractions, construct diagrams



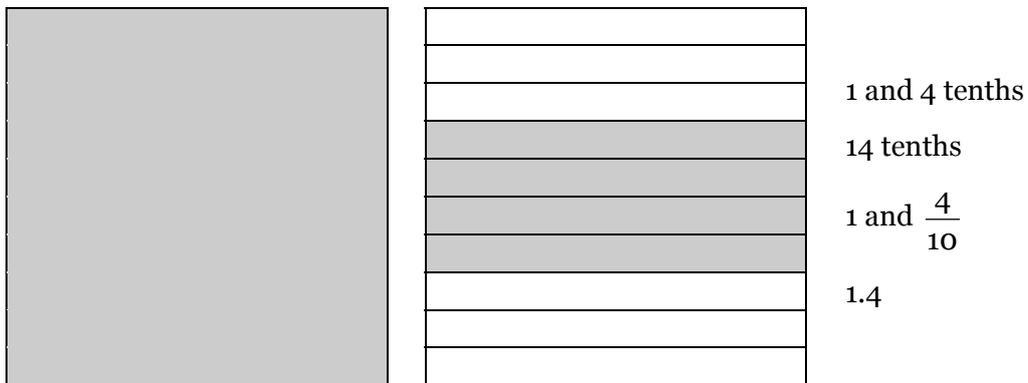
Develop ‘fifthing’ strategy by folding, observing and similar reasoning:



5. Introduce the fraction symbol:

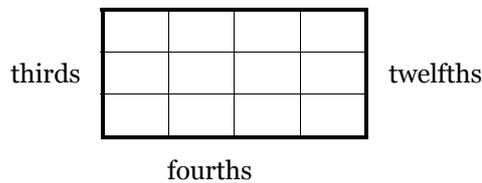


6. Introduce tenths via diagrams. Make and name ones and tenths, introduce decimal recording as a new place-value part.

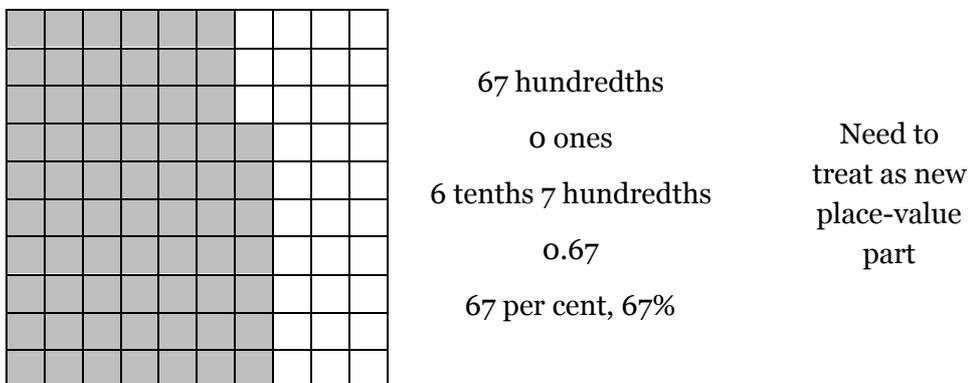


Treat as new place-value part. That is,

- (a) establish new unit, 10 tenths is 1 whole, 1 whole is 10 tenths;
 - (b) make, name and record ones and tenths, pointing out the need for a marker to show where the ones begin; and
 - (c) consolidate understanding through comparing, ordering, counting forwards and backwards in place-value parts, and renaming.
7. Extend partitioning techniques to develop understanding that thirds by fourths produce twelfths, tenths by tenths give hundredths and so on.



8. Extend decimal fraction knowledge to hundredths using diagrams and metric relationships, introduce percentage.



9. Explore fraction renaming (equivalent fractions) using paper-folding, diagrams, fraction trays (fraction wall) and games.

Establish the generalisation that if the number of parts (denominator) increases by a certain factor then the number of parts required (numerator) increases by the same factor

10. Link thousandths to metric relationships. Rename measures (grams to kilograms etc). Use partitioning strategies to show where decimals live.

e.g. 4.376 lives between 4 and 5... partition into tenths... it lives between 4.3 and 4.4... apply metaphor of a magnifying glass to 'stretch' out line between 4.3 and 4.4, partition into ten parts to show hundredths... it lives between 4.37 and 4.38... repeat process to show thousandths and identify where 4.376 lives.

11. Introduce addition and subtraction of decimals and simple fractions to support place-value ideas, extend to multiplication and division by a whole number.

$$\begin{array}{r} 4.26 \\ + 7.38 \\ \hline \end{array} \qquad \begin{array}{r} 5\frac{2}{3} \\ - 3\frac{5}{8} \\ \hline \end{array}$$

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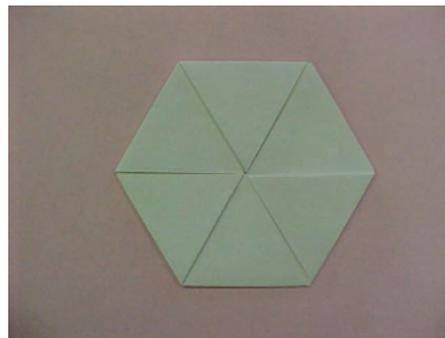
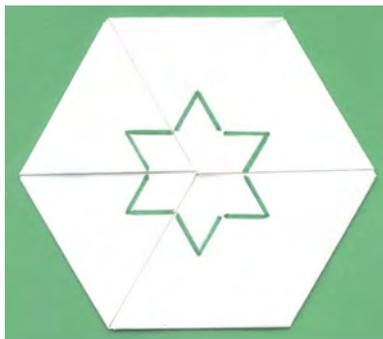
The hexaflexagon

Rob Smith & Gordon Cowling

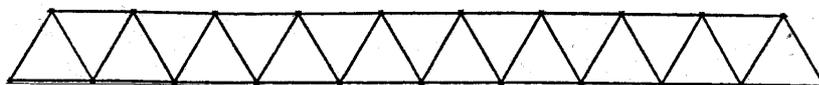
Gippsland Grammar, Qld

The first part of the session will focus on simply making and having fun with a hexaflexagon. We will make the hexahexaflexagon first, and learn how to 'flex' these little curiosities. In the middle part of the session we will make some other members of the hexaflexagon family: the trihexaflexagon and the tetrahexaflexagon and look at the templates for the next members of the family. Finally, we will look at 'Traverses', especially the 'Tuckerman Traverse' and then the secret of the hexaflexagon will be revealed. The beauty and simplicity behind such complicated looking templates will amaze you. Before you leave, you will be given a commercially made hexaflexagon to construct. This is a session where you get to use scissors, paste and coloured pencils (all provided).

A hexaflexagon is the name given to a folded paper model that, not surprisingly, is in the shape of hexagon. The fully constructed model looks like the following:



The fascinating thing about these models is that you start with a straight strip of paper, which is made up of a string of equilateral triangle.



The instructions for constructing the hexahexaflexagon are given below. By the way, there is a terminology associated with hexaflexagons. The basic hexaflexagon is the trihexaflexagon, which when flexed presents 3 different faces. The hexahexaflexagon displays six different faces. There is a whole family of hexaflexagons, which produce different numbers of faces.

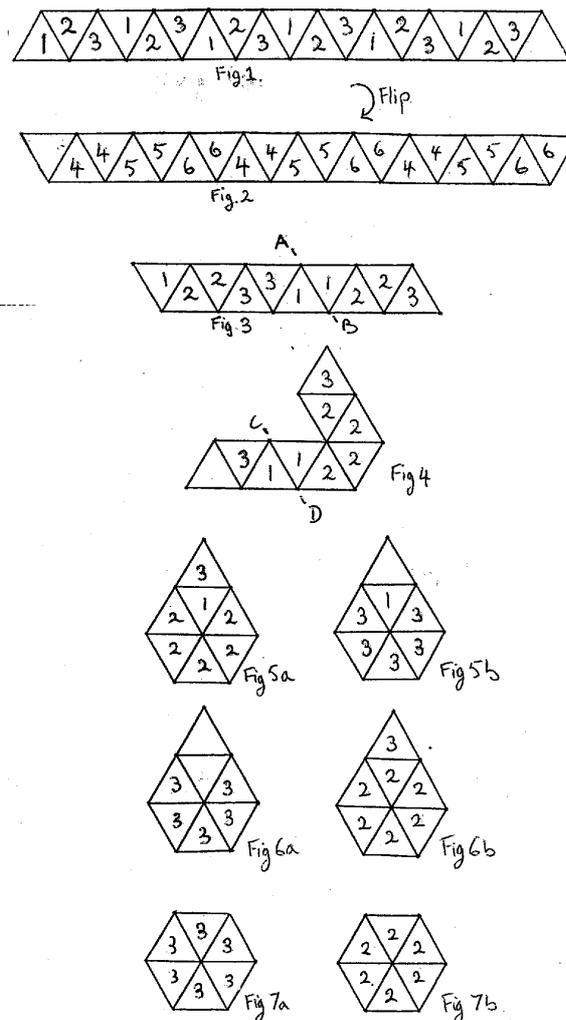
The hexahexaflexagon

To make this model, firstly get your strip of paper with 19 triangles along it. Number them carefully as shown in Figure 1 of the diagram.

Fold sharp creases along each of the lines joining the triangles.

Then starting with Figure 2, fold 4 onto 4, 5 onto 5, 6 onto 6, and keep repeating, until you have the shorter strip Figure 3.

Then fold 3 onto 3 to obtain Figure 5. One side has mostly 2s, while the other side has mostly 3s. Twist the ends, which will bring the last 2 to join the others and if you turn it around there will be mostly triangles with 3s in them and two blank triangles. Glue these two blank triangles together to finish your hexahexaflexagon.



The fun begins when you start flexing your hexaflexagon. The technique is unusual at first, but with practice becomes quite simple and in fact almost therapeutic!

The essence is to hold the flexagon in your left hand and then place the middle finger of your right hand behind a join. Place your index finger and thumb of your right hand above the join, which effectively give you a good grip of the flexagon in your right hand.

You now pinch this join together so that Figure 2 of the sequence is obtained.

With your left hand you now push in on the join directly opposite the one that you pinched together. This creates the three pointed star as in Figure 3. With your thumb and forefinger of your right hand holding the pinch together, and your left forefinger pushing in on the other side, now place your left thumb at the top of the star and prise the star open as in Figure 4.

This brings about a big surprise, as a new face will be revealed.

Playing with these models is fascinating as the patterns that are produced are great fun.

However, the more you see of them, the more their geometry becomes interesting, and the other members of the family need to be examined.

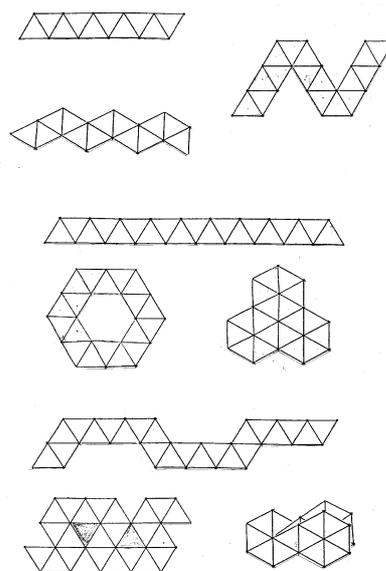
The question that struck me when I saw the templates for the other members of the family for the first time was simply how on earth did someone come up with them.

The top one is the trihexaflexagon, the one below it is the tetrahexaflexagon, the top right is the pentahexaflexagon. The long strip is the one just made — the hexahexaflexagon. Beneath it are two other templates for a hexahexaflexagon, while the bottom three are different templates for making a heptahexaflexagon.

Why did they look like this, and why were there no others? There seems to be no pattern at all.

I read the article in Martin Gardner's book, *Mathematical Puzzles and Diversions*. I also scoured the internet, but I invariably found that the author either knew no more than me, or seemed to assume that it was so obvious that it did not warrant explanation. Finally, I saw the pattern. It is so simple that I wonder it took me so long to see it.

Perhaps you have already seen it yourself, just from what I have described. If not, come along to the session and have it all revealed.



Spinners for beginners

Ed Staples

Erindale College, ACT

A *spinner* is a wonderful *Excel* spreadsheet tool for enhancing student learning. I do not believe that their existence is that well known, although I saw them demonstrated brilliantly by James Taylor at the MANSW conference in Leura last year. This workshop will focus on using the spinner in a variety of relevant applications including sketching, probability distributions, reducible interest, and Bezier curves. In the second half-hour, teachers in pairs will go through a simple hand on exercise on sketching where they will begin to develop the skill of using the spinner.

The use of spinners for computer assisted teaching is clear, however their real power lies in their simplicity and therefore become accessible to students in terms of their own independent investigations.

Introduction

Have you ever wanted to show students the effect of increasing or decreasing coefficients of polynomial functions dynamically? Have you ever wanted to graphically demonstrate the effect of changing interest rates, loan amounts, payment periods and loan terms dynamically? Have you ever wanted to study the behaviour of the function $y=(x-a)^n$ close to $x=a$ for various rational values of n . Have you ever wanted to run a tangent around a polynomial function describing as you go the changing gradient? Have you ever wanted to show a particle moving back and forth in Simple Harmonic Motion varying amplitude and frequency, or a projectile fired across a computer screen at an angle, velocity and vertical height of your choosing? Have you ever wanted to describe the principle of superposition caused by the addition of two sinusoidal waves of slightly different frequencies? Have you ever wanted to show distances between points, angles of inclinations of lines, and ratio divisions of lines dynamically? Do you know what the Euler segment is? Do you know what a Bezier curve is?

Bringing the spinner to life

Do you realise that all of this and much more are possible on the *Excel* spreadsheet that sits on your desktop computer? The dynamic nature of relationships can be explored using a device known as a 'spinner' that sits under the View menu in the tool bar. You

can bring the animal to life by clicking and dragging into your document. It's a simple device, as devices go, in that it simply increments or decrements a nominated cell by an integer value (of your choosing) over and over again. The whole procedure is enacted by clicking with you right index finger on the left side of the mouse while the screen pointer is directly over the spinner. That sounds hard doesn't it?

But what then? Well, suppose we take that incrementing cell and put it amongst a bunch of formulae. As we click on the spinner the formulae are immediately pushed around. And if we link these formulae to graphs, we can push around graphs as well! Suddenly we can make things move and all sorts of mathematical concepts come alive. The 'what happens if...' question is answered with perfect visual precision. Understanding effect and structure is taken to a new level.

At first glance, spinners seem a little limited in that they increment in integral jumps with a lowest value of zero. This problem is easy to get around. In fact the challenge to do so is itself illuminating. Suppose we attach the spinner to the cell $A1$ with integer increments. If we dump the formula $=A1/10-10$ into the cell $A2$, then $A2$ starts to iterate in tenths from negative ten. This method of transformation opens up the rational field. We need not stop there.

Some examples

Suppose we attach spinners to each of the three coefficients of a quadratic function, which is in turn attached to its graph. Along the way we determine the roots, the y -intercept and the discriminant of the associated quadratic function. As the spinner does it work we can wield the parabola across the screen, making it just touch the x -axis, or push it below the x -axis or even flip it upside down. We can watch the discriminant change from a negative value to zero and on to a positive value. All of the connections are made.

This powerful tool has been sitting in *Excel* all this time, and yet a lot of us have missed it. Instead we have jumped at much more sophisticated packages such as *Maple* and *Derive* which, despite their incredible power, lack the simplicity and adaptability of this little tool. In fact its attractiveness lies in its simplicity. I can at a moments notice create a program which focuses on one small concept I wish to introduce. And it happens all the time.

Just last week, I found my self de-seasonalising rainfall data with a group of Year 12 mid level students. Attaching spinners to the data illuminated the de-seasonalising concept. In August of 2002, it occurred to me that spinners could be used to flesh out the various effects of score standardisation. Consequently I had students of a Year 11 mathematics class playing with an appropriate *Excel* program with copious interconnected and interrelated spinners. A copy of this program has been included in the disc of examples distributed at the workshop. One aspect that emerged from this program was the counter-intuitive notion that the following two sets of scores produce the same standardisation:

100%, 1%, 1%, 1%, 1%, 1%, 1%, 1%, 1%, 1%

2%, 1%, 1%, 1%, 1%, 1%, 1%, 1%, 1%, 1%

The Z-scores for the two higher scores are both 3, independent of the scores themselves. In general, the value of the two Z-scores on a single set of $m+n$ scores of which exactly m of them have the value $a\%$ and n of them have the lower value $b\%$, are simply $\sqrt{\frac{m}{n}}$ and $-\sqrt{\frac{n}{m}}$. Of course I hadn't realised this until I began investigating spinners!

In another example, I had recently explained to an Extension 2 Year 12 NSW Higher School Certificate student where to find spinners and how to use them. A couple of weeks later, he showed me how he had used spinners to investigate the relationship given by $\tan \theta = \frac{v^2}{rg}$. Some of you might recall that this relationship describes the angle required to eliminate any sideways frictional force on a banked circular track (for a given speed and a given radius of curvature). The student showed me how he had used spinners to iterate the radius and plot θ against v . He noticed that the curve seemed to have an inflection at 30° irrespective of the changing radius. He then proceeded to algebraically find the inflections for the function $\theta = \tan^{-1}\left(\frac{v^2}{k}\right)$ and found that the angle was always 30° and independent of k .

Advantages of spinners

Perhaps I could say of spinners that:

1. They are adaptable to a range of concepts. Graphing packages can be expensive, and time consuming, and while perhaps not as cosmetic, spinners can be adapted to a variety of formulae whether graphed or not.
2. They are hands on tool (autonomy of approach/creativity). Nothing is prefabricated, and therefore we have the ability to focus in on a particular aspect of a concept.
3. They are adaptable to assignments. If your school has a student LAN, investigations of variability can be tailored to assignments that students access on line. The spinner can be used to focus students into a particular area.
4. They are useful to students in their own independent research. They are very easy to learn, and they can be of great benefit to high ability students.
5. They allow students to see movement through rapid iteration and in doing so enhance understanding of effects.
6. They can even be used to individualise assignments and mark responses quickly. We have looked at creating assignments to middle to low ability students on topics such as finance and trigonometry that assessed the same concepts but contained different numbers. It protected authenticity and yet was easy to mark.
7. They sit on the *Excel* spreadsheets. They are inexpensive and are readily accessible.

How do you access them?

In the *Excel* spreadsheet, click from the tool bar VIEW > TOOLS > FORMS. Click on the spinner and drag to anywhere in the work sheet. Right click on the spinner and click FORMAT CONTROL. In CELL LINK, put the cell reference A1 (or any other if you like!). Click OK, then click off the spinner, and then begin clicking the spinner up and down. It is as simple as that!

You will notice that the increments are in single units to create non-negative integers only; but do not let this put you off them because with the appropriate translation and dilation, we can increment to any rational number we choose. For instance we may wish to increment by 0.5 down wards. In cell A2, put the formula $=-0.5*A1$. Click the spinner and see the cell A2 decrease by 0.5 indefinitely.

From here, the sky is the limit.

A little taste of a spinner at work

Interrogating the linear function $y = mx + c$.

You are planning a lesson to show students how the graph of the line varies with different values of m and c . Open up *Excel* and create the moving graph.

1. Putting the headings in.

In cell A1 and A5 write the words 'gradient' and 'y-intercept'.

2. Create two spinners and cell link these to C1 and C5.

Remember to find the spinner: in the tool bar: View > Toolbars > Forms.

Select the spinner, and drag into spreadsheet. Right click to Format Control and Cell Link to the appropriate cell. (In our example, C1 or C5). Check to see that they are incrementing properly. Size each spinner so that it fits over the cells C1-C2-C3 and C4-C5-C6. Now move each spinner over those cells covering up the incrementing numbers.

4. Create two columns of x and y values.

For the x values, write x in the cell A9. In A10 put the number -10. In A11 put the equation: $=A10+1$.

Copy down to A30 by clicking and dragging the bottom right hand corner of highlighted cell A11.

For the y values, write y in the cell B9. In B10 put the equation: $=B\$1*A10+B\5 .

Copy down to B30 using the same technique.

At this stage the y values should all be 0 because cells B1 and B5 have nothing in them. Remember our spinners are linked to the cells C1 and C5. We did this because *we can only spin positive integers* with spinners. This is only a minor setback, because we know how to translate and dilate!

In cell B1 put the formula $=10-C1/5$.

In cell B5 put the formula = C5-10.

The effect is immediate. Clicking the spinners reveals positive and negative values of the gradient changing in tenths. Can you see why?

The y values in the column are changing as well.

5. The graph.

Now highlight the block of cells A10 to B30.

Click the Graph icon on the tool bar.

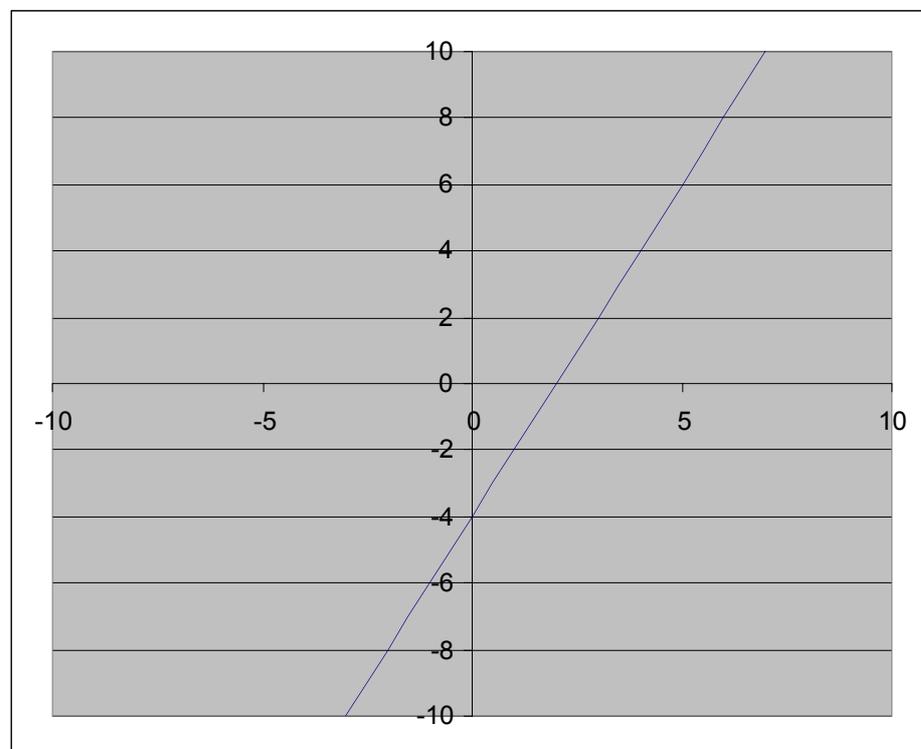
Click XY (Scatter) and choose the middle right option.

When the graph appears click and drag to the top of the spreadsheet. Size the graph appropriately. Click on the legend and press the delete key (I find it annoying!)

Start spinning to see the effect of changing gradients and y -intercepts.

You will notice that the axes are inclined to change scale. Before we fix this, click out of the graph. We can stop this by right clicking on each axes and choosing Format axes. *This has to be done for each axis separately.* Under the scale menu, we can stop the scale being automatically adjusted and replace with specific domain and ranges.

For the x -axis, choose -10 to 10, and for the y -axis choose -10 to 10.



Voila!

Lessons from variation research II: For the classroom

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This workshop will feature various activities devised as part of a research project studying students' understanding of variation and their improved outcomes following a chance and data unit emphasising variation. The activities arose from interview protocols and survey items used with students, from lessons taught by project teachers, or from professional development sessions led by the author with teachers. Each will be presented with a curriculum mapping associated with the topics in the chance and data curriculum. The stacked dot plot (or line plot) will be introduced as a straightforward means of representing data and displaying variation.

Students' appreciation of variation as the basis of all activity within the chance and data part of the mathematics curriculum has been the focus of research in Tasmania for the past several years. Some of the outcomes related to students' understanding are reported in Watson (2003), whereas classroom activities used in or derived from the project are reported here. Two principles underlie the suggestions that are made: first is that without variation there would be no chance and data curriculum and hence it must be the focus of much of the discussion; and second is that lesson planning in the curriculum should be investigation-based with the realisation that all parts of the chance and data curriculum should be addressed during the activities. The second principle is derived from the observation that in the early days of the chance and data curriculum, activities with dice or cards were used to fill in time on Friday afternoons with no objectives other than to 'do chance' and keep students occupied.

Four 'lessons' are presented here although they may be found to be appropriate for use over several class periods. The presentation will be appropriate for middle school, to be adapted for grades 3 to 9, depending on the class and curriculum objectives. Each activity is accompanied by a Curriculum Mapping created in the spirit of the second principle above. A Curriculum Mapping is *not* a lesson plan and teachers will need to plan at a level appropriate for their students, keeping in mind the various components of the curriculum. The main headings of the Curriculum Mapping reflect those of the *National Statement* (Australian Education Council [AEC], 1991) and *Profile* (AEC, 1994) and can be traced back to the seminal work of Peter Holmes (1980). After setting

a question the steps in an investigation involve (1) data collection, (2) data representation, (3) summarising data, (4) chance (depending on the question), and (5) drawing a conclusion. Sometimes more sophisticated language like ‘sampling’, ‘data reduction’, ‘probability’, and ‘inference’ are used and this may depend on the level of the class. Notice that the word ‘variation’ does not warrant a heading of its own. That is because variation is present at every point of an investigation and permeates all discussion that takes place.

The activities have been drawn and adapted from other sources, and used in teaching or interviews as part of research and/or as professional development sessions for middle school teachers (e.g. Watson & Kelly, 2002a). The main graphical form to be used in the activities is the stacked dot (or line) plot. Although it was introduced at the high school level by Landwehr and Watkins in 1986 and at the primary school level by Russell and Corwin (1989), it has only recently gained notoriety, for example featuring in the latest United States National Council of Teachers of Mathematics’ *Standards* (2000). The term ‘line’ plot comes from the simple placement of dots or Xs along a scaled line for some type of measurement. Figure 2 shows plots for the first suggested activity. Since the term ‘line graph’ is usually used for graphical representations with data connected by lines to show a trend, the more visual terminology, stacked dot plot, is used here.

Spinners

The first activity has been used in several forms for teaching and research (e.g. Torok, 2000; Watson & Kelly, 2002b). The form presented in the Curriculum Mapping in Figure 1 focuses specifically on the variation that occurs in repeated ‘samples’ of 40 spins of a 50-50 spinner (half black and half white). There is an expectation that out of 40 spins, 20 will land on black, but there is also the intuition that it might not be ‘exactly 20’ but ‘about 20’. How many different from 20 is reasonable? Is ‘40 black’ reasonable? Surprising?

In collecting data to investigate this question, students must decide on a method of sampling: issues include a consistent way to spin the spinner, perhaps a minimum number of circuits of the spinner, what to do with ‘liners’, and how to keep track of results. Students often work in pairs to collect data on how many blacks occur in 40 spins. Depending on the age of students the activity may also include collecting data for a ‘quarter’ spinner; that is, one that is $\frac{1}{4}$ black and $\frac{3}{4}$ white. Doing these data collections at the same time provides the opportunity to discuss two types of variation: variation due to the model represented in the spinner and variation due to the random process of spinning.

Although each individual set of 40 trials of the 50-50 spinner provides a point of discussion, for example whether exactly 20 blacks are obtained and if not how far from 20 the outcome is, the main interest is in producing the distribution of outcomes obtained by the class. Stacked dot plots for the outcomes of 40 spins for 25 students for a 50-50 and a ‘quarter’ spinner are shown in Figure 2. In recording results like these on a whiteboard it is important to draw the base lines so they are vertically aligned for easy comparison. Students should either record their results themselves or read them out

individually for the teacher to record. Watching and discussing how the distribution evolves as new values are added is part of learning about appropriate variation. Outliers are extreme values and it would be up to the class to decide how extreme a value is before it is discarded — reasons for such action must be stated, such as ‘the spinner was warped’. Recording the outcomes for the ‘quarter’ spinner after the 50-50 spinner affords the opportunity to compare distributions and particularly to discuss values such as ‘15 blacks’, which are possible and reasonable for both types of spinner. The possible overlap of a few values in the two distributions illustrates why it is important to perform many sets of 40 spins and look at the overall shape of the distribution obtained.

Curriculum mapping for ‘Spinner’ investigation

Setting the question
 What sort of variation occurs in many outcomes from spinning a ‘half’ spinner?
 A ‘quarter’ spinner?

Chance/probability

1. What are the ‘chances’ of success for each of the spinners?
2. What is the expected number of outcomes from 40 spins for each type of spinner?
3. What is reasonable variation from this number?

Data collection (count data)

1. Spin a ‘half’ spinner 40 times and record how many times it lands on the designated ‘half’.
2. Spin a ‘quarter’ spinner 40 times and record how many times it lands on the designated ‘quarter’.

Representing data

1. Produce a class line plot to combine individual results for the ‘half’ spinner (scale 0 to 40).
2. Produce a class line plot to combine individual results for the ‘quarter’ spinner above or below the first with the same scale.

Summarising data

1. Middles, compare and contrast.
2. Ranges, compare and contrast.
3. Outliers, compare and contrast.
4. Shape, compare and contrast.
5. Compare data with expected outcomes.

Drawing conclusions
 Write a report on the similarities and differences in outcomes for the two types of spinners when they are spun many times. What sort of distributions [shapes] of outcomes occur?

Figure 1. Spinner investigation.

The degree to which formal statistics, such as the arithmetic mean, are used to summarise the information in the stacked dot plots, depends on the grade level and associated curriculum. The mode (most frequent outcome) and the median (middle value on the plot) are adequate for most purposes to describe and distinguish the two

distributions in Figure 2. Discussion of the shape, even in colloquial descriptive terms, is important. What does it make you think of? 'A forest.' 'A hill.' 'A train.'

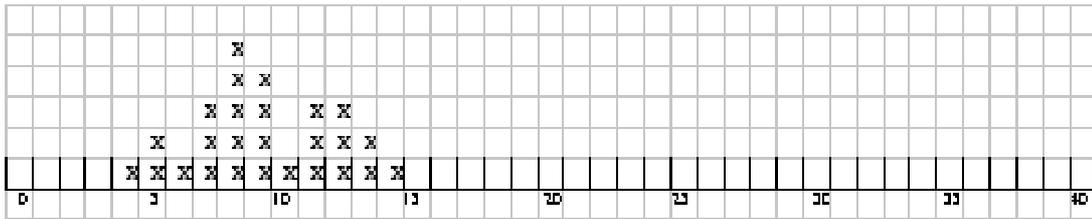
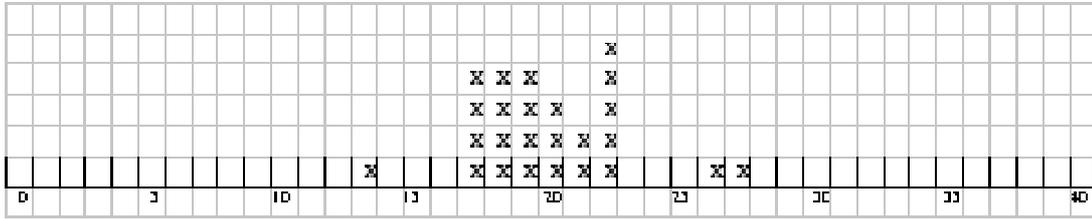


Figure 2. Outcomes for 25 sets of 40 spins for a 'half' (top) and a 'quarter' (bottom) spinner.

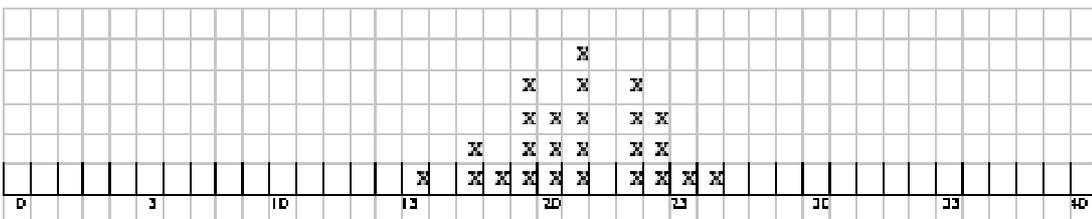
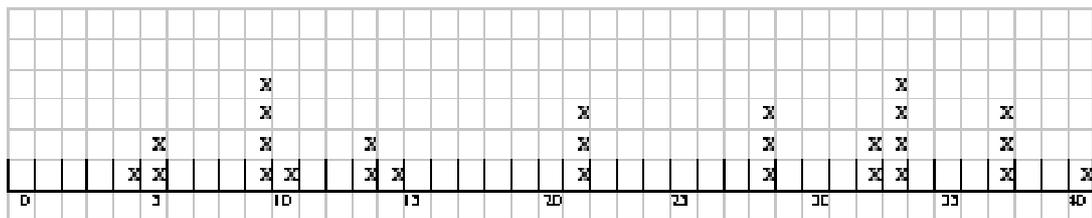
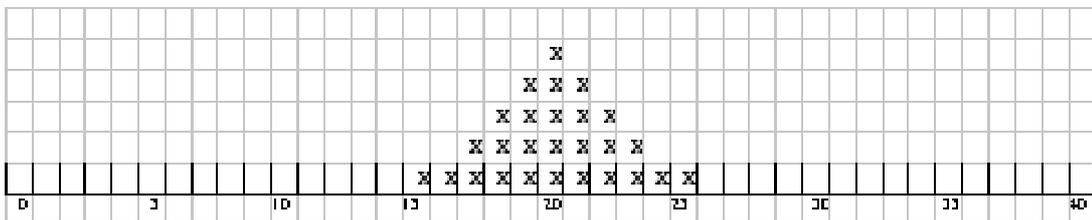


Figure 3. Which graphs are authentic?

For this activity the link to chance is natural since the probabilities of $\frac{1}{2}$ and $\frac{1}{4}$ associated with the black areas of the spinner translate to expected numbers of 20 and 10 black outcomes, respectively for the two spinners, in 40 trials. In discussing what is reasonable, students can be asked to consider outcomes from three ‘other’ classes shown in Figure 3 and determine which they think are authentic and which are made up. The final, and very important part of the activity, is to write up a report on the investigation. Each student should write of their own experiences with the spinners, as well as the class outcomes, and describe what a reasonable distribution of outcomes should look like. This should be rich in the language of variation.

Measurement

There are many opportunities to make connections between measurement and data handling in the mathematics curriculum. Since measurements usually vary in one way or another, measurement activities are a natural vehicle for studying variation. Three related activities are suggested in the Curriculum Mappings in Figures 4, 5, and 7, based on the measurement of students’ hand spans and foot lengths. These measurements (variables) are chosen because they provide student ownership of the data without being threatening to self-esteem, as weight and height might be.

Curriculum mapping for ‘Measurement of hand span’ investigation

Setting the question
 What is the *typical* hand span measurement of grade X students?

Data collection (interval data)

1. Each student measures hand span (spread out, to nearest .5 cm).
2. Discuss how many measurements would be needed for a good estimate.

Representing data

1. List and order class data.
2. Create a line plot for class data.
3. Create a stem-and-leaf plot.

Summarising data

1. Mean, median, mode, range.
2. Box-and-whisker plot.

Chance
 Discuss ways of randomly choosing a sample for this problem.
 How would chance affect the mean, etc?

Drawing a conclusion
 Write a report, including all of the assumptions made, to explain how the class arrived at a typical hand span for grade X.

Figure 4. Hand span investigation.

Notice that the question for the investigation is stated in terms of ‘typical’ hand span, not ‘average’ hand span. This is done on purpose to avoid preconceived ideas of average and encourage discussion of what a typical measurement would be. Before starting to collect data, students should decide what group (population) they are representing: their class, their grade across the school, their grade across the state. They should also consider whether the sample size represented by their class is reasonable to obtain a good estimate. Again before collecting data, rules need to be established to be confident that the data are comparable. Several possibilities exist for hand span measurements and the accuracy to which they are made. Students should contribute ideas and agree on the method. Popular alternatives in middle school include measuring the spread from thumb tip to little finger tip extended as far as possible along a ruler to the nearest 0.5 cm. This provides a reasonable scale and spread for most classes.

Listing and ordering the class measurements can be a messy affair without some forethought. Creating a stacked dot plot on the board with students placing post-it labels with their recorded measurements in the appropriate location is a useful activity, especially to reinforce decimal scaling. The data are thus ordered and could be moved to an ordered list or other representation. A stem-and-leaf plot (the stems being the whole numbers, say 15 to 23 and the leaves being either 0 or .5 depending on the measurement) can be created. This representation is likely to appear more clustered than the stacked dot plot with its scale measured in 0.5 cm. The fact that it will tell a similar story but with a different visual orientation warrants mention. Again discussion of variation, outliers, and the shape of the graph is necessary.

In summarising the data, certainly the median, mode, and range are appropriate. The relationship of the mode to the colloquial meaning of typical is a point to be made. Is the mode sufficient to describe the distribution of measurements? Introducing the mean is not essential unless part of the curriculum for the grade level. For some grades this activity provides a good opportunity to introduce the box-and-whisker plot. Based on ‘middles’, drawing a box-and-whisker plot only requires finding the median (middle) of the entire data set and then the middle of the lower half and the middle of the upper half. These can be determined by counting along the Xs in the stacked dot plot, splitting the difference if an even number of values is involved. A box then is drawn enclosing values in the middle half of the data set with whiskers extending to the extreme values. Although easy to draw, it takes some skill to explain the variation displayed; for example, the smaller half of the box denotes more closely packed data values, whereas the longer the whisker, the more spread in the extreme values. Box-and-whisker plots are shown for the data sets in Figure 6.

The place of chance in this investigation depends on the grade level. Randomly choosing samples with a chance device and plotting their values with changing summary values will illustrate how a random procedure can effect the distributions obtained. Detailed discussion and experimentation leading to the Central Limit Theorem is an appropriate extension for higher grades. Again writing a report is an essential part of the investigation. This should be assessed for clarity and completeness, with the expectation of improvement in future investigations.

An extension of the investigation in Figure 4 is given in Figure 5, where the question becomes one of whether the typical hand span is longer for girls or boys. The

comparison of two data sets is now recommended for middle school students (NCTM, 2000, p. 50) and can be very motivating. Again hand span is a variable that should be non-threatening in this regard. It also introduces a second variable to discuss, gender. Gender is a categorical variable and should be contrasted with the measurement variable associated with hand span. Data can be displayed in a back-to-back stem-and-leaf plot with easy visual comparison of the two groups. Variation may be different, as may be the range or outliers.

Curriculum mapping for ‘Comparing measurements on two groups’ investigation

Setting the question

Do boys have greater hand spans than girls?

Data collection (interval data)

Measurement of hand span.

Data representation

1. List and order class data by gender.
2. Create line plots for each group.
3. Create a stem-and-leaf plots for each group; put these ‘back-to-back’.

Summarising

1. Compare shapes of distributions in line plots and stem-and-leaf plots.
2. Compare means, medians, modes.
3. Create and compare box-and-whisker plots for the two groups.

[**Chance/probability** – in higher grades could test differences in means to see if ‘significant’.]

Drawing a conclusion

Write a report answering the question and explaining the analysis carried out to reach the conclusion.

Figure 5. Comparing two groups.

There are several other ways to compare the distribution of hand spans for the two groups, e.g. using different symbols on the same stacked dot plot, placing two plots aligned vertically, or placing the two sets of data above and below the base line. Box-and-whisker plots can be added and used as a further basis for describing the difference in variation for the two sets. Figure 6 contains stacked dot plots and box-and-whisker plots for two genuine adult data sets for 10 males and 10 females. Note that for a small data set, one half of a box may collapse in a box-and-whisker plot.

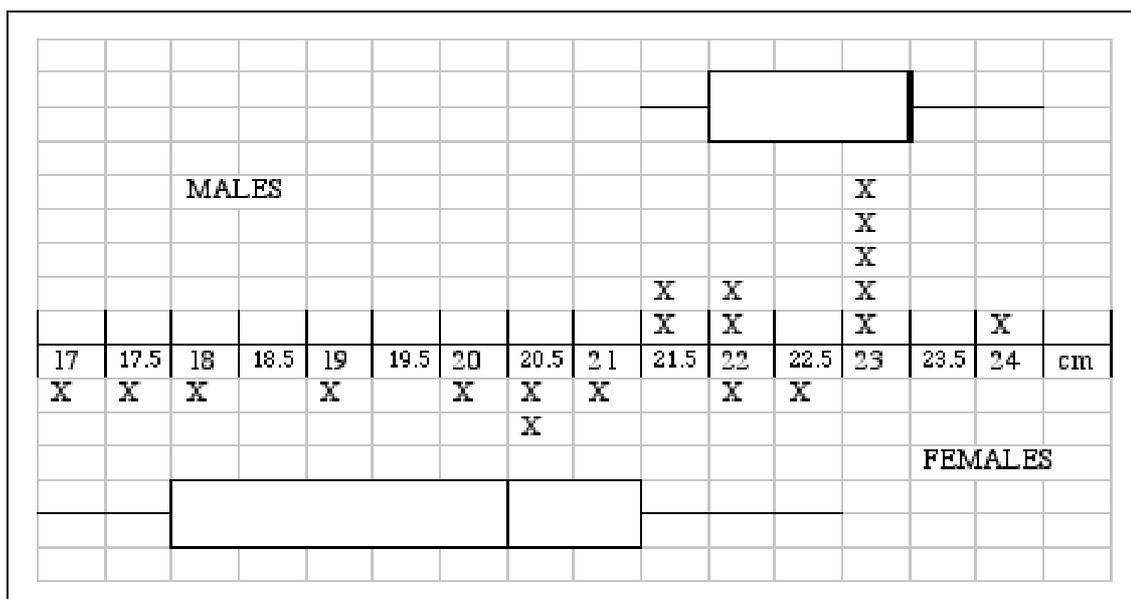


Figure 6. Comparing male and female hand spans.

The Curriculum Mapping in Figure 7 introduces a second measurement variable, foot length, and asks a question about the association of hand span and foot length. If it is planned to progress to this activity then it might be useful to do all data collection initially. It is certainly important for students to retain their hand span measurements. The issue of cause-and-effect can also be addressed in an investigation such as this. Does a long foot length cause a long hand span? Does some other variable cause them both?

Curriculum mapping for 'Association of two variables' investigation

Setting the question

Is there an association between a person's hand span and foot length?

Is there a 'cause' of the association?

Data collection (interval data)

1. Each student measures foot length (shoe-off — to nearest .5 cm).
2. If not done before, measure hand span.

Data representation

1. List data in pairs for all students (perhaps use a spreadsheet).
2. Have each student put 2 measurements on a yellow post-it and put up on a graph on the wall.
3. Each student to produce a graph on paper.

Summarising data

1. Find mean of data on each axis, mark point.
2. Discuss outliers.
3. Find a 'line of best fit' — use ruler or 3-medians approach.

[**Chance/probability** — in higher grades could find correlation on calculator and see if 'significant']

Drawing a conclusion

Write a report answering the question and explaining how the analysis was carried out. Speculate on the 'cause'.

Figure 7. Association of two variables.

Data representation is an important feature of this investigation because some students will not have encountered a scattergram before. Using post-it notes can be very valuable here. Each student records name, hand span (cm) and foot length (cm) on the post-it and then determines its position on a large class scattergram. Name is important here because it is the *person* who determines the point on the graph: both measurements are made on that person. There is one point (person) determined by two measurements (hand span, foot length). If students have difficulty with this idea a useful activity is to go onto the school basketball court, draw and label axes, and have students actually stand where their point is on the grid. For some, actually spreading their arms at right angles pointing to the values of their hand spans and foot lengths on the axes is good further reinforcement of the relationships between the point and its determining coordinates.

The scattergram should display both a pattern associated with the expected relationship of increasing foot length with increasing hand span and variation association with expected individual differences. Both of these are important to discuss as some students expect a perfectly straight line relationship to be shown. If there are no apparent outliers in the data set, students can be asked to imagine where the swimmer Ian Thorpe would be placed on the graph.

Conclusion

These four investigations illustrate various types of chance and data activities that can be used in investigations. Data may be represented as counts (number of black spins), as measurements (hand span or foot length) or categories (gender). All offer opportunities to observe, display, summarise, and draw conclusions about variation in data sets. Such activities are motivating because they are complete. They cover all aspects of the curriculum without having to fill in work sheets or draw graphs of meaningless data. They also integrate literacy into the mathematics curriculum.

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