Partitioning — The missing link in building fraction knowledge and confidence

Dianne Siemon

RMIT University (Bundoora), Vic.

This paper describes and justifies partitioning as the missing link between the intuitive fraction ideas displayed in early childhood and the more generalised ideas needed to work with rational number more formally in the middle years of schooling.

Part 1 — Fractions and the middle years

Part 2 — Partitioning — the key to formalising fraction ideas

Part 3 — Steps in formalising Fraction ideas in the Middle Years

Part 4 — Operations involving decimals and fractions

Part 1 — Fractions and the middle years

The *Middle Years Numeracy Research Project* (MYNRP), conducted in a structured sample of Victorian Primary and Secondary schools from November 1999 to November 2000, used relatively open-ended, 'rich assessment' tasks to measure the numeracy performance of approximately 7000 students in Years 5 to 9. The tasks valued mathematical content knowledge as well as strategic and contextual knowledge and generally allowed all learners to make a start.

The project found that the major factor affecting overall performance was the differential performance on tasks concerned with the use of rational number. 'Hotspots' identified by the analysis of the data indicated that a significant number of students in Years 5 to 9 have difficulty with some or all of the following.

- Explaining and justifying their mathematical thinking;
- Reading, renaming, ordering, interpreting and applying common fractions, particularly those greater than 1;
- Reading, renaming, ordering, interpreting and applying decimal fractions in context;
- Recognising the applicability of ratio and proportion and justifying this mathematically in terms of fractions, percentage or written ratios;
- Generalising simple number patterns and applying the generalisation to solve a related problem;
- Working with formulae and solving multiple steps problems;
- Writing mathematically correct statements using recognised symbols and conventions;

- Connecting the results of calculations to the realities of the situation, interpreting results in context, and checking the meaningfulness of conclusions; and
- Maintaining their levels of performance over the transition years from primary to secondary school.

These findings have important implications for the teaching and learning of fractions and related number ideas in the middle years of schooling. Once thought to be unnecessary apart from further mathematics study, this area of the curriculum is now recognised as a key contributor to what it means to be numerate in that it underpins the important notion of proportion on which so much of our everyday life depends.

Why fractions?

It is no longer acceptable that students leave school without the foundation knowledge, skills and dispositions they need to be able to function effectively in modern society. This includes the ability to read, interpret and act upon a much larger range of texts than those encountered by previous generations. In an analysis of commonly encountered texts, that is, texts that at least one member of a household might need to, want to, or have to deal with on a daily, weekly, monthly or annual basis, approximately 90% were identified as requiring some degree of quantitative and/or spatial reasoning. Of these texts, the mathematical knowledge most commonly required was some understanding of rational number and proportional reasoning, that is, fractions, decimals, percent, ratio and proportion. An ability to deal with a wide range of texts requires more than literacy – it requires a genuine understanding of key underpinning ideas and a capacity to read, interpret and use a variety of symbolic, spatial and quantitative texts. This capacity is a core component of what it means to be numerate.

Understanding the problem

Even before they come to school many young children exhibit an awareness of fraction names such as half and quarter. During the first years of schooling, most will be able to halve a piece of paper, identify 3 quarters of an orange and talk about parts of recognised wholes (e.g., blocks of chocolate, pizza, Smarties etc.). While to an adult ear, this sounds like children understand the relationships inherent in fraction representations, for many they are simply using these terms to describe and/or enumerate well-known objects. Such children may not be aware of or even attending to the key ideas involved in a more general understanding of fractions; that is, that equal parts are involved, the number of parts names the parts, and that as the number of parts of a given whole are increased, the size of the parts (or shares) gets smaller.

The use of fraction words to name recognised parts of recognised wholes lulls adults (teachers and parents) into thinking that many of these children are able to understand and use the fraction symbol in the early years of schooling despite the fact that most curriculum advice now advocates a delay in the formalisation of fractions. When children are introduced to the fraction symbol without a deep understanding of what each part of the symbol refers to they are inclined, quite naturally, to view both the numerator and the denominator as counting or 'how many' numbers. This leads ultimately to such misconceptions as, '$\frac{3}{12}$ is bigger than $\frac{3}{8}$ because 12 is bigger than 8'. It also leads to the well-known 'Freshman's Error', that is the tendency to add denominators when adding fractions.

Even when students start to work with practical activities aimed at elaborating the meaning of fractional parts in the middle years of primary school, there is no guarantee that the 'equal-ness' of the parts is necessarily attended to. For instance, although chocolate blocks and pizzas are partitioned into equal parts by virtue of their manufacture or the method of cutting, this could be overlooked if each piece is treated as a one and counted. Similarly, a packet of Smarties may be shared out equally but in doing so the relationship of the parts to the whole is no longer obvious. This issue is evident in student responses to fraction diagrams as well. For instance, when given the following diagram and asked to show 2 fifths, many children will simply count and colour without recognising the fractional relationship between the parts and the whole.

Shade or colour to show $\frac{2}{5}$.

Given a diagram with 6 equal parts and asked to show 2 fifths, some students appeared not to notice the number of parts or assumed that the teacher had made a mistake and coloured in 2 parts regardless. A number of students also coloured in two parts even when they were given a diagram with five unequal parts. For anyone who understands fractions and how such diagrams are conventionally read, this activity is not a problem, indeed it is trivial. But as teachers we tend to assume that students read the diagrams in the same way we do. This is not necessarily so. Initially, students attend to the lines and the shapes inherent in fraction diagrams, which is not surprising given their experience of working with 2-D shapes over four to five years of schooling. They are not necessarily focussing on area or indeed the relationship between one particular shape and another.

It is my very strong view that colouring in someone else's fraction diagram is next to useless in scaffolding young students' thinking about fractions. To understand the point of this task, students need to understand how such diagrams are constructed and read. But first, examples and non-examples of fraction representations need to be explored to ensure that students recognise that equal parts/equal shares are necessary. There are many ways to do this, for instance, marking plasticene rolls into equal and unequal parts, sharing the packet of Smarties equally and unequally to distinguish quarters from 4 parts, and talking about the implications of having the netball court divided up into 3 unequal parts.

Another important initial idea is the distinction between 'how many' (the numerator) and 'how much' (the denominator). This can be supported by distinguishing between the number of parts (numeral) and the size of the parts (name) when practical fraction examples are first encountered, for example, the Goal Shooter can play in 1 *third* of the netball court, Jason ate 3 *quarters* of the pizza. Ultimately, students need to understand that 3 fifths means not only 3 'out of' 5 equal parts, but 3 divided by 5 and that this is a number that exists uniquely on the number line irrespective of how it is named (for example, $\frac{3}{5}$, 0.6, 60%, or $\frac{15}{25}$). However, before this can be realised, students first need to understand how parts are formed, named and renamed.

In the past, this step has been omitted. Teachers and mathematics programs have tended to assume that once students can identify fractions from a given diagram, shade a given diagram to show a given fraction (nearly always a proper fraction), or find a simple part of a given whole (e.g., $\frac{1}{2}$ 25%, or $\frac{1}{3}$ of 24), they are familiar with fractions and ready to proceed to renaming fractions (equivalent fractions) and performing more complex operations on fractions. However, if we are to prevent students adopting narrow, rule-based approaches to fraction manipulation in later years we must revisit how fractions are formalised in the middle years, paying careful attention to what I believe is the missing link, that is, the connection between fractions and partitive division, and thereby to multiplicative thinking more generally. As suggested above, this begins with a deeper understanding of how fractions are made, named and renamed — in other words, partitioning.

Click here to join the discussion on the AAMT Bulletin Board

Part 2 — Partitioning: the key to formalising fraction ideas in the middle years

By partitioning, I mean not just the experience of physically dividing continuous and discrete wholes into equal parts but generalising that experience to enable students to create their own fraction diagrams and representations on a number line by applying a range of well-known partitioning strategies. Three strategies, applied singly or in combination, appear to be sufficient to achieve this, namely, *halving, thirding* and *fifthing*.

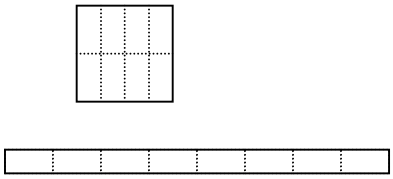
Halving, thirding and fifthing

Students do not need to be taught how to halve. This is an intuitive process that most students are familiar with. Successive halving yields all of the fractions in the halving family, that is, halves, quarters, eighths, sixteenths etc. Students should be encouraged to explore halving with different wholes (e.g., coloured squares, newspaper, paper streamers, plasticine, rope), noting similarities and differences and recording observations and generalisations such as:

- as the number of parts increase they get smaller;
- apart from halves and quarters, the number of parts names the parts — for example, 3 equal parts: thirds; 5 equal parts: fifths.

Similarly, successive thirding generates all of the fractions in the thirding family, that is, thirds, ninths, twenty-sevenths etc and successive fifthing generates all the fractions in the fifthing family, that is, fifths, twenty-fifths and so on. By combining strategies students can investigate what fractions can be generated, for example, sixths, twelfths and eighteenths can be generated by halving and thirding, and tenths, twentieths and hundredths can be generated by halving and fifthing.

Experiments with paper folding (using different paper sizes and shapes as well as paper streamers) support the following thinking for the *halving* strategy.

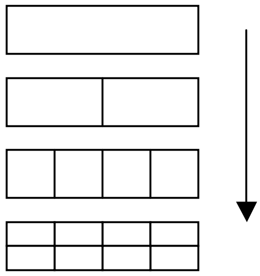


Paper Models

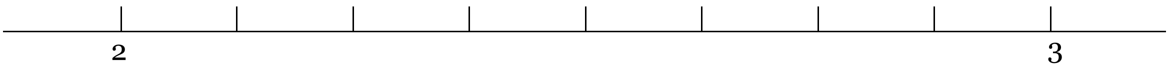
What did the first fold do? It cut the top and bottom edge in half – estimate and connect.

What did the next fold do? It cut the halves in half again – estimate and connect.

What did the final fold do? It cut the sides in half – estimate and connect.

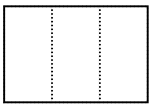


Region Diagrams

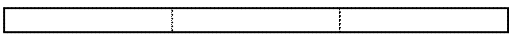


Number Line Representations

For *thirding*, the thinking strategy can be described as follows.



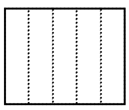
Paper Models



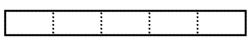
Is 1 third bigger or smaller than 1 half? It's smaller – estimate, leaving space for 2 more parts of the same size, connect to show 1 third

What needs to be done with remaining part? Halve it to create 2 more parts the same size

For *fifthing*, the thinking can be described as follows.



Paper Models

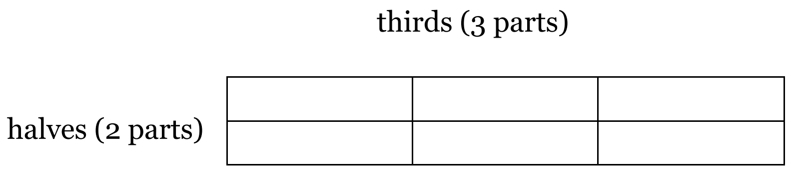


Is 1 fifth bigger or smaller than 1 quarter? It's smaller – estimate and connect to show 1 fifth.

What needs to be done next? Halve and halve again to create 4 more parts the same size

Students also need to recognise that for fraction models involving area, two parts may look different but have the same relationship to the whole. For example, a square piece of paper can be folded in half to form a triangle or a rectangle. Although the halves are different shapes, they are both halves of the same whole and are therefore the same in at least one important respect, area. To ensure that students are attending to area as the relevant dimension, they need to explore different ways of making the same fractional parts, for example, ninths can be made by partitioning each side of a rectangle into 3 equal parts or by successively partitioning one side into 9 equal parts.

A well developed capacity to partition regions and lines into any number of equal parts supports fraction renaming and justifies the use of multiplication in this process. For instance, the recognition that region diagrams may be partitioned on two sides, leads to the observation that thirds (3 parts) by halves (2 parts) gives sixths (6 parts). This demonstrates the link to the region or area model of multiplication and supports further generalisations based on this idea; in particular, the generalisation supporting fraction renaming: *where the number of parts is increased (or decreased) by a certain factor, the number of parts required is increased (or decreased) by the same factor*.



This eliminates the need for, and the problems caused by the inappropriate rule, 'what you do to the top you do to the bottom', as students have the capacity, through partitioning, to identify what is happening to the number of parts. For example, for 2/5 students can see that as the number of parts *in* the whole is increased by a factor of 2 (or doubled) to 10 parts, that the number of parts *of* the whole has also increased from 2 to 4, to show 4/10

Click here to join the discussion on the AAMT Bulletin Board

Part 3 — Steps in formalising fraction ideas in the middle years

Before undertaking this activity, it is suggested that teachers review a range of familiar examples to ensure that students understand the difference between continuous and discrete fraction situations, for example, 2 thirds of a length of rope as opposed to 2 thirds of 24 apples.

The first three components are essentially review and consolidation

1. Review initial fraction language and ideas by discussing 'real-world', every-day examples involving continuous and discrete fractions.

Continuous

e.g. 2 and 3 quarter pizzas

e.g. 2 thirds of the netball court

Discrete

e.g. half the grade to art, half to the library

e.g. 2 out of 12 eggs are cracked

2. Practice naming and recording (not symbols) every-day fractions using oral and written language, distinguishing between the count (how many) and the part (how much) and including mixed as well as proper fractions.

   e.g. 3 fifths, 3 out of 5 equal parts;

   2 wholes and 3 quarters (fourths)

3. Use practical examples and non-examples to ensure foundation ideas are in place, that is,

   - recognition of the necessity for equal parts or fair shares and an appreciation of part-whole relationships (e.g., half of this whole may be different to half of that whole) – fractions are essentially about proportion;
   - recognition of the relationship between the number of equal parts and the name of the parts (denominator idea), particularly the use of ordinal number names; and
   - an understanding of how equal parts are counted or enumerated (numerator idea).

4. Introduce the 'missing link' — PARTITIONING — to support the making and naming of simple common fractions and an awareness that *the larger the number of parts, the smaller they are.*

   Begin by making links to what students already know, for example, their knowledge of the numbers 0 to 100 and the respective location of these numbers.

   > Rope Activity – you will need a length of rope (at least 3 – 4 metres long), some clothes pegs and some cards or paper (quarter A4 is good). Invite 2 students to hold the ends of the rope with cards labelled 0 and 100. Distribute cards and pegs to other students and ask them to peg the numbers (e.g., 3, 19, 48, 67, or 92) on to the line. Discuss strategies to demonstrate the practical use of well-known fractions, e.g., 'it's about half', 'I know that 67% is about 2 thirds of the way'

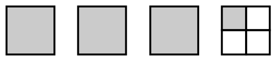
   Many students have a good sense of percent from their lived experience. This can be used as a springboard for partitioning as it helps build a sense of proportion.

   > Percent Mentals (Source: Shelley Dole, Catholic Education Office, Melbourne, 2003) – Invite students to record answers only to questions such as, 50% of 20, 50% of 350, 25% of 80, 40% of 150, $33\frac{1}{3}$% of 12, $33\frac{1}{3}$% of 60, $12\frac{1}{2}$ 1/2% of 40 … Discuss strategies, particularly the use of well-known fraction equivalents.

   Use 'kindergarten squares', scrap paper, and paper streamers to investigate *halving*. Explore and teach strategies for *thirding* and *fifthing* derived from paper folding/rope experiments and estimation based on reasoning about the size of the parts (see discussion above).

   Use materials to make and name fractions, exploring what fractions can be made by combining partitioning strategies, e.g., twelfths can be made by combining halving and thirding.

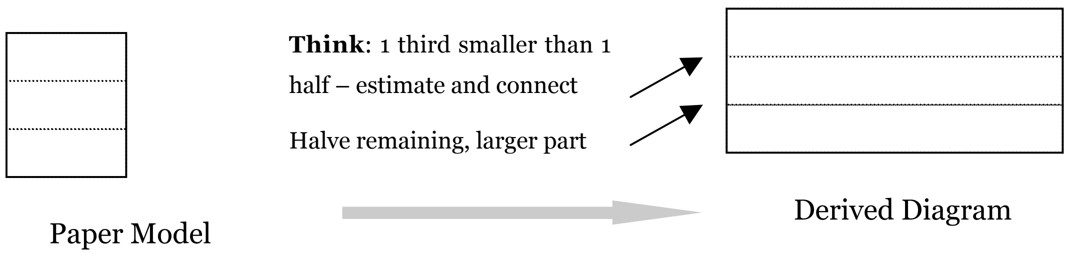
Poster Activity - Involve groups of students in making posters about particular fractions (e.g., 3 and 1 quarter, 5 eighths, 2 and 4 fifths, 2 ninths etc). Students make fractions using coloured paper squares (white on one side) then write as much as they can about their fraction, e.g., it's bigger than 3 but smaller than 3 and a half. It can be renamed in terms of so many halves and quarters, or so many quarters or eighths etc.



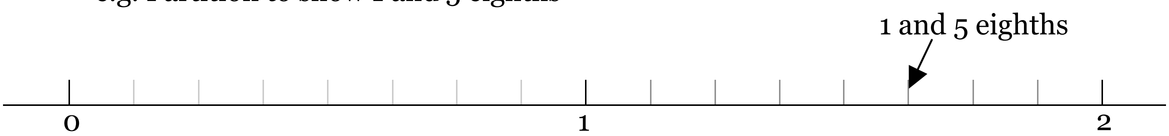
Other representations can be added to the posters as these are acquired, e.g., number line, real-world example etc.

Use partitioning strategies and thinking derived from paper folding to construct fraction diagrams and number line representations.

e.g. Use *thirding* strategy to partition the rectangle on the right into thirds



e.g. Partition to show 1 and 5 eighths



Think: Halve and halve again and halve again ...

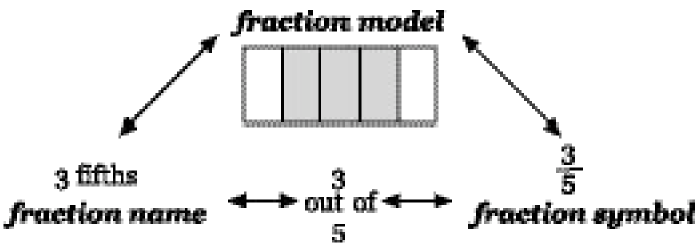
e.g. Use *fifthing* strategy to draw a diagram showing 2 and 2 fifths

Think: 1 fifth is smaller than 1 quarter, estimate 1 quarter then 1 fifth. What needs to be done next? Halve and halve again to make 4 more parts the same size ... repeat and shade as needed

Fraction Estimation (Source: Maths300, Curriculum Corporation) – this computer based resource can be used to consolidate students' sense of proportion and their partitioning skills. It provides immed

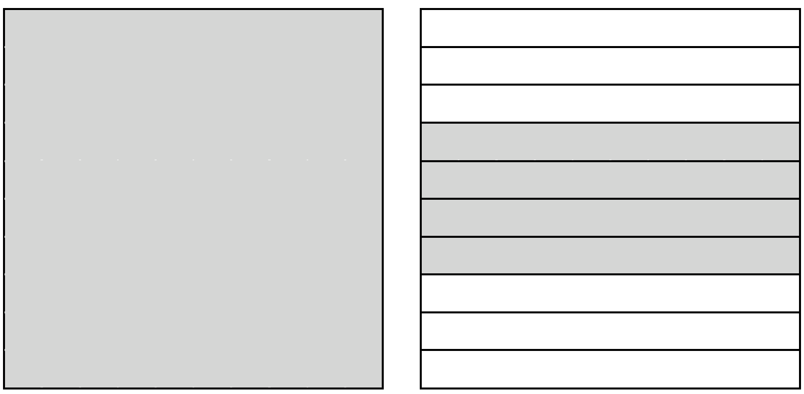
iate feedback in the form of % accuracy which also helps with percent sense .

5. Introduce (or revisit) the fraction symbol in terms of the ‘out of’ idea for proper fractions:



It is likely that many students will have already encountered the fraction symbol informally (perhaps even formally), but may not have been exposed to and understood the idea of partitioning. It is strongly recommended that irrespective of whether or not students have met the fraction symbol that they become familiar with partitioning as this equips them to make, name and rename fractions with understanding. It also underpins a more generalised understanding of proper and improper fractions in terms of division; e.g. $\frac{3}{5}$ not only means 3 ‘out of’ 5 equal parts or 3 fifths, but more generally, 3 divided by (or shared among) 5. This leads to the recognition that any number can be partitioned into any number of parts, written in fraction notation and simplified as necessary; e.g. $\frac{36}{9}$ means 36 divided by 9, $\frac{4}{5}$ means 4 divided by 5 and so on.

6. Introduce tenths via fraction diagrams and number line representations. Make and name ones and tenths using the *fifthing* and *halving* partitioning strategies (keeping in mind that zero ones is just one example of ones and tenths).

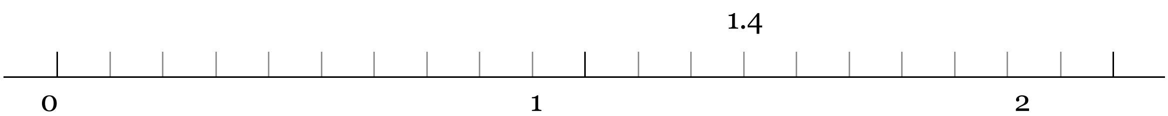


1 and 4 tenths

14 tenths

$1\frac{4}{10}$

1.4



Introduce decimal recording as a new place-value part. That is,

(a) establish the new unit, 10 tenths is 1 one, 1 one is 10 tenths (1 tenth of these is 1 of those);

(b) make, name and record ones and tenths, pointing out the need for a marker (the decimal point) to show where the ones begin; and

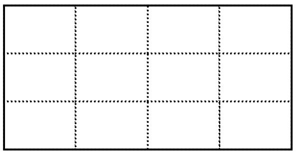
(c) consolidate understanding through comparing, ordering, counting forwards and backwards in place-value parts, and renaming.

Show the connection between common and decimal fractions and percent by doing the division and renaming; e.g. $\frac{4}{5}$ means 4 divided by 5, this cannot be done unless 4 ones is renamed as 40 tenths, 40 tenths divided by 5 is 8 tenths or 0.8 or 80 hundredths or 80%; for $\frac{3}{8}$, use a calculator to show that 3 divided by 8 is 0.375 or 37.5 hundredths, 37.5%. This is equivalent to trading to the hundredths place.

7. Extend partitioning techniques to develop understanding that thirds by fourths produce twelfths, tenths by tenths give hundredths and so on.

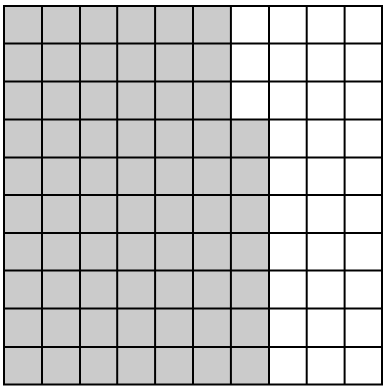
fourths (4 parts)

thirds (3 parts)



Thirds by fourths give twelfths (12 parts)

8. Extend decimal fraction knowledge to hundredths using diagrams (tenths by tenths), number line representations and metric relationships (money and MAB can lead to misconceptions), introduce percentage as another way of writing hundredths.



67/100

67 hundredths

0 ones 6 tenths 7 hundredths

0.67

67 per cent, 67%

Treat as new place-value part (as for tenths above)

Linear Arithmetic Rods (Source: Department of Science & Mathematics Education, The University of Melbourne) – These materials are made from very thin 20 mm washers (represent thousandths) and 20 mm plastic conduit tubing (variously cut to represent hundredths, tenths and ones) and can be used to demonstrate relative magnitude of decimal place-value parts.

Rope Activity – The length of rope referred to earlier can be used to locate common and decimal fractions. Try labelling the ends 0 and 2 respectively and have students peg on the numbers, 3/4, 0,8, 1.4, 2/5, and so on. This is enlightening as many students insist that 3/4 is located at 3 quarters of the length of the rope.

Target Practice – You need four ten-sided dice per group of up to six students. Target numbers can be written on the board or on a worksheet. Students take it in turns to throw all four dice. The object is to use three of the digits to make a number as close to the target number as possible. Target numbers can be varied to include whole and/or decimal numbers. A mental calculation is made to determine how close the number made is to the target number. This is repeated a number of times and the results summed. The winner is the person with the lowest sum.

9. Explore fraction renaming (equivalent fractions) using paper-folding, diagrams, and games.

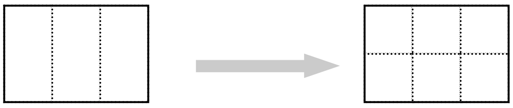
Make a Whole – this is a board game using a fraction wall (a one, halves, thirds etc up to fifteenths), a set of numbered cards to 10 and a set of fraction names (halves, thirds, quarters etc). Cards a turned face down and each student (or pair of students) takes it in turns to turn over 1 number card (e.g. 4) and 1 fraction name (e.g. thirds). This amount is outlined/shaded on the fraction wall as appropriate. Where thirds are already shaded, the turn can still be taken if an equivalent fraction can be found and shaded (this might involve more than one row of the fraction wall).

Establish the generalisation that if the number of parts (denominator) increases by a certain factor then the number of parts required (numerator) increases by the same factor in order to maintain equality (see earlier discussion).

10. Introduce thousandths in terms of metric relationships. Rename measures (grams to kilograms etc). Use partitioning strategies to show where decimals live. In particular, emphasise the relationship, 1 tenth of these is 1 of those.

e.g. 4.376 lives between 4 and 5… partition into tenths… it lives between 4.3 and 4.4… apply metaphor of a magnifying glass to 'stretch' out line between 4.3 and 4.4, partition into ten parts to show hundredths… it lives between 4.37 and 4.38… repeat process to show thousandths and identify where 4.376 lives.

The concepts associated with fractions and fraction recording need ultimately to be extended to rates and ratios. It is beyond the scope of this paper to explore this in detail but these build on the 'for each' or Cartesian product idea of multiplication, for example, 60 km/h, $1.75/kg. The idea of 'for each' can be illustrated by partitioning, for instance, having 'thirded' a piece of paper then halved it, it can be pointed out that for each third there are now 2 additional parts ... 3 by 2 parts, giving 6 parts or sixths.



Difficulties experienced with rates and ratios are more likely to be the result of naïve views of multiplication and/or lack of understandings about how fractions are made, named and renamed than it is with these notions per se.

Click here to join the discussion on the AAMT Bulletin Board

Part 4 — Operations involving decimals and fractions

The addition and subtraction of decimals and simple fractions can be introduced once these numbers are well understood in their symbolic form. Start with simple, like fractions, ones and tenths and related fractions. Recording should build on previous recording and support place-value ideas. It needs to be remembered that the only reason this is done is to consolidate fraction renaming. Written recording should not need to be taught – it should be obvious. If not, do not do it!

e.g.

4 fifths	2.3	$1\frac{1}{2}$
−2 fifths	+4.8	$+3\frac{3}{4}$

At this stage, no formal process should be used to rename fractions. Related fractions should be renamed mentally based on well-known equivalences, for example,

4 fifths take 2 fifths is 1 fifth. Record

3 tenths and 8 tenths, 11 tenths or 1 and 1 tenth. Record the tenths with the tenths and the 1 with the ones. 7 ones altogether

$\frac{1}{2}$ and $\frac{3}{4}$? Rename $\frac{1}{2}$ as $\frac{2}{4}$, 2 quarters and 3 quarters is 5 quarters or 1 and $\frac{1}{4}$. Record the parts with the parts and the 1 with the ones. 5 ones altogether

Later on, once the generalisation for fraction renaming is well understood, more complex examples can be introduced such as,

4.26	$5\frac{1}{3}$
+ 7.38	$-3\frac{5}{8}$

In this case, decimal addition (and subtraction) is fairly consistent with what has gone on before but the subtraction (or addition) of unlike fractions requires renaming and, in this case, decomposition of 1 one for 24/24.

e.g. $\frac{1}{3}$ take $\frac{5}{8}$? No, need to rename as like fractions [Think: eighths by thirds, twenty-fourths. Is there anything simpler? No, rename as twenty-fo

3 parts increased by a factor of 8, so $\frac{1}{3}$ = 8/24

9 parts increased by a factor of 3, so $\frac{5}{8}$ = 15/24

$\frac{8}{24}$ take $\frac{15}{24}$? No, need to trade 1 one for $\frac{24}{24}$

$\frac{32}{24}$ take $\frac{15}{24}$? Yes, $\frac{17}{24}$. Record with parts

4 take 3 is 1

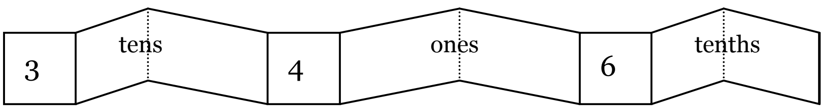
4 $^{32}/_{24}$
5 $^{1}/_{3}$ $^{8}/_{24}$
- 3 $^{5}/_{8}$ $^{15}/_{24}$
$1\ ^{17}/_{24}$

In my view, we do not need to dwell on these types of written calculations. They should only be introduced once students have a very firm grasp of what the fraction symbol means, and how fractions are made, named and renamed. If this is a problem — do not do it!

The same applies to the multiplication and division of fractions: if you need an answer to any calculation for the purposes of solving a problem, USE A CALCULATOR. The only reason written solutions are pursued is to emphasise the way in which operations work with various numbers and to develop conventions of mathematical literacy — still important even in this day and age as we need to be able to communicate what we are doing and why in a way that convinces others.

The multiplication and division of decimal fractions is fairly straightforward provided the appropriate concepts are understood. In this case, the area idea for multiplication and the partition idea for division (see Booker et al., 2003 for an elaboration of these ideas).

e.g. 34.6 × 7 can be supported by a Number Expander



x 7

e.g. 4.3 × 2.7 can be represented as 43 tenths by 27 tenths and treated as for whole number (to arrive at 1161). Tenths by tenths are hundredths so the product is 1161 hundredths or 11.61. This is far preferable to the meaningless rule, 'If there are 2 numbers after the decimal point in the question, then there will be 2 numbers after the decimal point in the answer'.

One further issue: when multiplying or dividing decimal fractions by powers of 10, the decimal point does not move. It lives between the ones and tenths – it is the digits that move. Taking decimal points on a jumping journey is a major source of students errors in relation to decimal operations and decimal place-value.

e.g. For 7.329 × 100, only have to think that 7 ones become 7 hundreds

7.329
732.9

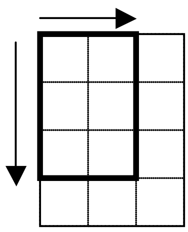
Partitioning has already introduced students to multiplication of fractions by fractions informally. Students can work with fraction diagrams that they create themselves to prompt the generalisation that applies for the multiplication of proper fractions, that is, *that the parts (denominators) multiply to give the new part and the number of parts (numerators) multiply to indicate how many of the new parts are required*

Partitioning builds on 'region' and 'area' models of multiplication. This leads to the 'by' or 'for each' idea and, more generally, the factor.factor.product view of multiplication and division which regards multiplication and division as inverse operations – this is the idea needed to support all further work in this area and algebra.

e.g. The partitioning idea sees $\frac{3}{4}\times\frac{2}{3}$ in terms of region or area, for instance if the following diagram represents 1 whole, the product can be thought of in terms of thirds by quarters … twelfths. 3 parts by 2 parts, 6 parts … $\frac{6}{12}$ or $\frac{1}{2}$

2 thirds

3 quarters



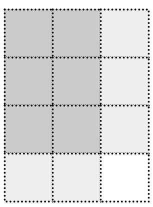
In this case, each side is partitioned to show $\frac{2}{3}$ and $\frac{3}{4}$ respectively, the product is the intersection of these two lengths. The area is named by the parts produced, that is, thirds by quarters … twelfths.

That is, $\frac{3}{4}$ **by** $\frac{2}{3}$ is $\frac{6}{12}$ …

What do you notice? Why?

Having said this, it needs to be noted that there are other ways of representing $\frac{3}{4}\times\frac{2}{3}$. For instance, it is possible, indeed, it is common practice in many texts, to use the more naïve 'fraction as operator' or the 'of' idea to justify the multiplication of a proper fraction by a proper fraction. This can be shown in the following diagram.

e.g. $\frac{3}{4}\times\frac{2}{3}$



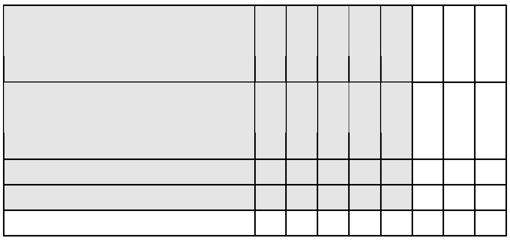
In this case, one of the fractions, generally the second, is shown ($\frac{2}{3}$), then the remaining side is partitioned into quarters and 3 are shaded. As this necessarily partitions the 2 thirds into quarters, the intersection shows $\frac{3}{4}$ **of** $\frac{2}{3}$ …

The problem with this is that the focus is on the number of parts, not how or why the new parts are created.

This representation of the 'of' idea only works for the multiplication of proper fractions, where the intersection remains a fraction of the original whole.

By contrast, the 'area' idea derived from partitioning supports all forms of fraction multiplication.

e.g. $2\frac{2}{3} \times 1\frac{5}{8}$



This is consistent with the area model used for whole number multiplication, where students learnt that each place-value part is multiplied by each other place-value part, for example, 45 × 23 can be modelled by MAB to show that the product is the area formed by (4 tens × 2 tens) + (4 tens × 3 ones) + (5 ones by 2 tens) + (5 ones by 3 ones).

This leads to the idea that, in this case, the mixed fraction multiplication is just as easily carried out as with renaming done mentally. Again, this would only be done where students had the necessary pre-requisite knowledge — if not, do not do it.

$$\begin{aligned}(2 \times 1) + \left(2 \times \frac{5}{8}\right) + \left(\frac{2}{3} \times 1\right) + \left(\frac{2}{3} \times \frac{5}{8}\right) &= 2 + 1\frac{1}{4} + \frac{2}{3} + \frac{10}{24} \\ &= 3 + \frac{6}{24} + \frac{16}{24} + \frac{10}{24} \\ &= 3\frac{32}{24} \\ &= 4\frac{8}{24} \\ &= 4\frac{1}{3}\end{aligned}$$

Division of a fraction by another fraction has always caused problems with many teachers resorting to the rule 'just invert and multiply — it works'. The division of a fraction by another fraction is largely problematic because students do not have access to the more sophisticated ideas of multiplication upon which it depends, in particular the 'factor.factor.product' idea. While quotation division (asking 'how many in') makes some sense for a whole number divided by a fraction and simple fraction situations, this does not generalise to all cases of fraction division.

e.g. For $6 \div \frac{1}{2}$ translated as 'How many halves in 6?' it is relatively easy to see that there are 12 halves in 6.

For $1\frac{1}{4} \div \frac{1}{4}$, it is also relatively easy to count quarters to get 5 quarters.

A much better way of doing this is to build on partition division which asks ‘how many in each part’ as opposed to ‘how many groups in’ (see Booker et al., 2003 for a detailed discussion of the difference between these two ideas). The partition idea leads naturally to fractions (hence, partitioning) and to the idea of ‘factor.factor.product’. It also leads to the strategy ‘think of multiplication’, for instance, 24 shared among 4 or partitioned into 4 equal parts gives rise to the thinking, ‘4 whats are 24?’.

This supports fraction division as well, for instance, $\frac{3}{4} \div \frac{2}{3}$ can be thought about as, ‘$\frac{2}{3}$ by what is $\frac{3}{4}$?’. This gives rise to the equation, $\frac{2}{3}x = \frac{3}{3}$ which can be solved by multiplying $\frac{3}{4}$ by $\frac{3}{2}$, which justifies (and is the source of) the rule, ‘invert and multiply’. Of course, students would not be doing this if they did not have access to the appropriate strategies for solving equations such as this, which require an understanding of multiplication and division in terms of ‘factor-factor-product’; but if they have not got access to these ideas and strategies, they should not be attempting fraction division or algebra in the first place.

Postscript: I am beginning to think that all division from Day 1 should be recorded as a fraction, but perhaps that’s a discussion for another day!

References

Booker, G., Bond, D. Briggs, J. & Davey, G. (1997). *Teaching Primary Mathematics*. Melbourne: Pearson.

Cahn, R. (1999). *Maths Fun Pack: Computer Games for Years 5–9*. Educational Software Productions.

Curriculum Corporation (2001). *Maths Project 300*. Web-based resource available from Curriculum Corporation (see http://www.curriculum.edu.au/maths300).

Graham, N. (1998). *Dice and Spinners*. Claire Publications: UK.

Lovitt, C. & Clarke, D. (1985). *Mathematics Curriculum and Teaching Program (MCTP) Activity Bank, Volumes I and II*. Melbourne: Curriculum Corporation.

Lovitt, C. & Lowe, I. (1993). *Chance & Data Investigations*. Melbourne: Curriculum Corporation.

Click here to join the discussion on the AAMT Bulletin Board